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FINITENESS CONDITIONS FOR PROJECTIVE AND INJECTIVE MODULES

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ABSTRACT. Does Hopkins' theorem extend to projective modules, i.e., are projective Artinian modules Noetherian? An example is given to answer this question in the negative; however, we show that the answer is affirmative for certain large classes of projective modules. Dually, are injective Noetherian modules Artinian? Again the answer is negative; nevertheless, we provide an affirmative answer for certain classes of injective modules.

Introduction. It is well known that the endomorphism ring of a module which is both Artinian and Noetherian is semiprimary. The author has noted [7] that the endomorphism rings of both projective Artinian and injective Noetherian modules are semiprimary. This raised the following questions: Do there exist projective Artinian modules which are not Noetherian, or could it be true that Hopkins' theorem [**11**, p. 132] extends to projective modules? Dually, do there exist injective Noetherian modules which are not Artinian [**8**], [**9**, p. 378]?

In §1 we prove that the endomorphism rings of both projective Artinian and injective Noetherian modules are semiprimary. We give an example of a projective Artinian non-Noetherian module in §2 and show that Hopkins' theorem does extend to a projective Artinian R-module M in each one of the following cases: (a) R is commutative, (b) R is hereditary, or (c) M is a generator in the category of R-modules.

An example of an injective Noetherian non-Artinian module is given in [**14**]. In §3 we prove that if M is an injective Noetherian R-module where R is a ring with polynomial identity which satisfies the ascending chain condition on annihilators of submodules of M, then M contains an essential submodule which is Artinian. From this it follows that injective Noetherian modules over commutative rings are Artinian.

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Throughout this paper, R will denote an associative ring which does have a unity and M will denote a unital right R-module. If S is a subset of M, then $r(s)$ denotes $\{x \in R: Sx=0\}$.

We note that the results in this paper can be immediately obtained for certain abelian categories. See [**10**].

1. **Semiprimary endomorphism rings.** The following theorem was announced in Fisher [**7**].

THEOREM 1.1. *If M is a projective Artinian R-module, then $S=\mathrm{End}_R(M)$ is semiprimary. Dually, if M is an injective Noetherian R-module, then $S=\mathrm{End}_R(M)$ is semiprimary.*

PROOF. We will prove only the first statement since the proof of the second follows immediately by dualizing the proof we give. First we show that S satisfies the descending chain condition on principal right ideals. Suppose that $\phi_1 S \supseteq \phi_2 S \supseteq \phi_3 S \supseteq \cdots$ is a descending chain of principal right ideals in S. For each n, there exists $s_n \in S$ such that $\phi_n s_n = \phi_{n+1}$. Thus $\phi_n M \supseteq \phi_n s_n M = \phi_{n+1} M$ and so we obtain the following descending chain of R-submodules of M: $\phi_1 M \supseteq \phi_2 M \supseteq \phi_3 M \supseteq \cdots$. Since M is Artinian, there exists a k such that $\phi_k M = \phi_{k+1} M = \phi_{k+2} M = \cdots$. We claim that $\phi_k S = \phi_{k+1} S$. Consider the following diagram.

$$\begin{tikzcd} & M \arrow[d, "\phi_k"] & \\ M \arrow[r, "\phi_{k+1}"] & \phi_{k+1} M \arrow[r] & 0 \end{tikzcd}$$

Since M is projective, there exists $\psi: M \to M$ such that $\phi_{k+1}\psi = \phi_k$. Thus $\phi_k S = \phi_{k+1} S$ and so S satisfies the descending chain condition on principal right ideals.

By Bass [**2**, Theorem P] we have the Jacobson radical, $J(S)$, of S is left T-nilpotent and $S/J(S)$ is semisimple Artinian. It follows from Fisher [**8**, Theorem 1.5] that $J(S)$ is nilpotent. Therefore S is semiprimary.

REMARK 1. Theorem 1 remains true if "projective" is replaced by "quasi-projective" and "injective" is replaced by "quasi-injective". Also M need not be unital.

REMARK 2. That M injective Noetherian implies $\mathrm{End}(M)$ is semiprimary is due to Fisher and Small and appears with another proof in [**8**]. Recently the author discovered that M projective Artinian implies $\mathrm{End}(M)$ is semiprimary is due to Harada and appears with different proof in [**10**]. Furthermore, Harada uses this to prove that M is finitely generated and, quite surprisingly, that $\mathrm{End}(M)$ is right Artinian. We take this opportunity to inflict on the reader a direct proof, which does not rely on $\mathrm{End}(M)$ being semiprimary, that M is finitely generated.

LEMMA. *If M is a projective Artinian R-module, then M is finitely generated.*

PROOF. First it can be shown that the Jacobson radical, $J(M)$, of M is small in M by using a technique of proof used by Kasch-Mares [**13**, Satz] and Miyashita [**15**, Proposition 3.4]. As is well-known, M Artinian implies that $M/J(M)$ is a finite direct sum of simple modules. Hence there exists $m_1, m_2, \cdots, m_k$ in M such that $m_1R+m_2R+\cdots+m_kR+J(M)=M$. However, $J(M)$ is small in M. Thus $m_1R+m_2R+\cdots+m_kR=M$ and so M is finitely generated.

REMARK 3. After establishing that M projective Artinian implies M is finitely generated, one can also obtain that $\mathrm{End}(M)$ is right Artinian by applying Sandomierski [**18**, Corollary 1].

2. **Projective Artinian modules.** We begin with an example of a projective Artinian module which is not Noetherian.

EXAMPLE. As in Fisher [**6**, Example 4], let V be a countably infinite dimensional vector space over a field F and let $\{v_1, v_2, \cdots\}$ be a basis for V. For each positive integer i, consider the subspace V_i of V which is spanned by $\{v_1, v_2, \cdots, v_i\}$. Define the linear transformation T on V by $v_1T=0$ and $v_iT=v_{i-1}$ for $i\geqq 2$. Let $F[T]$ be the polynomial ring generated over F by the transformation T. By considering V as a right $F[T]$-module, we see that the proper $F[T]$-submodules of V are precisely the V_i, $i=1, 2, \cdots$. Hence V is an Artinian non-Noetherian $F[T]$-module.

Form $R=\left(\begin{smallmatrix} F & V \\ 0 & F[T]\end{smallmatrix}\right)$ and set $M=\left(\begin{smallmatrix} F & V \\ 0 & 0\end{smallmatrix}\right)$. Then M is a projective right R-module since it is a direct summand of R. Also it is easy to show that each proper submodule N of M is of the form $\left(\begin{smallmatrix} 0 & W \\ 0 & 0\end{smallmatrix}\right)$ where W is a nonzero $F[T]$-submodule of V. Therefore M is a cyclic projective Artinian non-Noetherian R-module.

We proceed with some positive results. Let M be a projective R-module with trace ideal T. An R-module N is called *T-accessible* if and only if $NT=N$. See [**18**].

THEOREM 2.1. *If M is a projective Artinian R-module, then M satisfies the ascending chain condition on T-accessible submodules.*

PROOF. $\mathrm{End}(M)$ is right Artinian by Harada [**10**, Theorem 2.8]. Then Hopkins' theorem [**11**, p. 132] guarantees that $\mathrm{End}(M)$ is also right Noetherian. By the Lemma, M is a finitely generated projective. Hence Sandomierski [**18**, Corollary 1] applies to show that M satisfies the ascending chain condition of T-accessible submodules.

Now we consider cases in which Hopkins' theorem extends to projective modules.

THEOREM 2.2. *Let M be a projective Artinian R-module. If* (a) *R is commutative*, (b) *R is right hereditary, or* (c) *M is a generator for the category of right R-modules, then M is a Noetherian R-module.*

PROOF. (a) By the Lemma, M is finitely generated. Now M Artinian implies that $R/\imath(M)$ is Artinian by Northcott [**16**, Theorem 2, p. 180]. Then Hopkins' theorem [**11**] forces $R/\imath(M)$ to be Noetherian. Again by Northcott [**16**, Theorem 2] we obtain that M is Noetherian.

(b) If R is right hereditary, then each submodule of M is projective by Cartan-Eilenberg [**4**, Theorem 5.4]. Wherefore, by the Lemma, each submodule of M is finitely generated. Therefore M is Noetherian.

(c) This is implicit in Harada [**10**, Theorem 2.5]. Alternately, since each submodule of M is T-accessible by Sandomierski [**18**, Proposition 1.2], it follows that M is Noetherian by Theorem 2.1.

REMARK 4. The proof of (a) can be obtained from (c) by using Cartan-Eilenberg [**4**, Proposition 6.1, p. 30] and Bass [**3**, Proposition 4.7, p. 70].

3. **Injective Noetherian modules.** Miller and Turnidge [**14**] have produced an example of an injective Noetherian module which is not Artinian. From the following theorem it will result that certain classes of injective Noetherian modules are Artinian. We say that R is a *P.I.-ring* if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

THEOREM 3.1. *If M is an injective Noetherian R-module where R is a P.I.-ring which satisfies the ascending chain condition on annihilators of submodules of M, then M contains an essential submodule which is Artinian.*

PROOF. Since M is injective Noetherian, it follows by standard techniques that M is a direct sum of finitely many indecomposable injective Noetherian R-modules. Hence it suffices to assume that M is indecomposable. There exists an ideal P which is a maximal element in $\{\imath(N): N$ is a nonzero submodule of $M\}$. Moreover, P is prime and there exists $e \in M$ such that $P=\imath(eR)$. If the right ideal $\imath(e)^*=\imath(e)/P$ in the prime P.I.-ring $R^*=R/P$ is essential, then it contains a nonzero two-sided ideal by Amitsur [**1**, Theorem 9]. This contradicts the fact that $\imath(eR)$ is the largest two-sided ideal contained in $\imath(e)$. Thus $\imath(e)^*$ is not essential in R^* and so there exists a nonzero right ideal U^* such that U^* is R^*-isomorphic to $(U+\imath(e))/\imath(e)$. Hence U^* is R^*-isomorphic to an R^*-submodule of eR. Since eR is a uniform R^*-module, the R^*-injective envelope of U^*, denoted $E_{R^*}(U^*)$, is R^*-isomorphic to $E_{R^*}(eR)$. Because $E_{R^*}(eR)$ is an essential extension of eR as R-modules, we have that $E_{R^*}(eR)$ is

R-isomorphic to an R-submodule S of M. Now R^*-submodules of $E_{R^*}(eR)$ are R-submodules and hence $E_{R^*}(U^*)$ is a Noetherian R^*-module.

Since R^* is a prime P.I.-ring, it has a simple Artinian two-sided classical quotient ring Q by Posner's theorem [**17**]. Furthermore, $E_{R^*}(R^*)$ coincides with Q and is homogeneous, i.e., it is a finite direct sum of isomorphic injective indecomposable R^*-submodules. Each of these isomorphic injective indecomposable R^*-submodules of Q is R^*-isomorphic to $E_{R^*}(U^*)$ which is a Noetherian R^*-module. From this is follows that $Q=E_{R^*}(R^*)=R^*$ by Faith-Walker [**5**, Lemma 2.1]. Wherefore $E_{R^*}(R^*)$ is an Artinian R^*-module. Consequently $E_{R^*}(eR)$ is an Artinian R^*-module and hence S is an Artinian R-module which is essential. This completes the proof.

COROLLARY 3.2. *If M is an injective Noetherian R-module where R is a right hereditary Noetherian P.I.-ring, then M is an Artinian R-module.*

PROOF. Let N be a maximal Artinian submodule of M. Since R is right hereditary, M/N is an injective Noetherian R-module. If $N\neq M$, then Theorem 3.1 yields a nonzero Artinian R-submodule of M/N. This contradicts the choice of N. Therefore $N=M$ and M is Artinian.

COROLLARY 3.3. *If M is an injective Noetherian R-module with R commutative, then M is an Artinian R-module.*

PROOF. Again we may assume that M is indecomposable. We have that M is an injective $R/\imath(M)$-module by Kaplansky [**12**, Theorem 203] and $R/\imath(M)$ is Noetherian by Northcott [**16**, Theorem 2, p. 180]. We consider M as an $R/\imath(M)$-module and let P be as in the proof of Theorem 3.1. As is well known in the commutative case M is primary with primary radical P. As we have shown, P is maximal since R^* is its own quotient field. Moreover, P is nilpotent since $R/\imath(M)$ is Noetherian. It is now clear that M is Artinian.

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REFERENCES

1. S. A. Amitsur, *Prime rings having polynomial identities with arbitrary coefficients*, Proc. London. Math. Soc. (3) **17** (1967), 470–486. MR **36** #209.

2. H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

3. ———, *Algebraic K-theory*, Benjamin, New York, 1968. MR **40** #2736.

4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956. MR **17**, 1040.

5. C. Faith and E. A. Walker, *Direct-sum representations of injective modules*, J. Algebra **5** (1967), 203–221. MR **34** #7575.

6. J. W. Fisher, *Decomposition theories for modules*, Trans. Amer. Math. Soc. **145** (1969), 241–269. MR **40** #5656.

7. ———, *Endomorphism rings of modules*, Notices Amer. Math. Soc. **18** (1971), 619–620. Abstract #71T-A85.

8. ———, *Nil subrings of endomorphism rings of modules*, Proc. Amer. Math. Soc. **34** (1972), 75–78. MR **45** #1960.

9. R. Gordon, *Ring theory*, Academic Press, New York, 1972.

10. M. Harada, *On semi-simple abelian categories*, Osaka J. Math. **7** (1970), 89–95. MR **42** #7748.

11. I. Kaplansky, *Fields and rings*, Univ. of Chicago Press, Chicago, Ill., 1969. MR **42** #4345.

12. ———, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR **40** #7234.

13. F. Kasch and E. A. Mares, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J. **27** (1966), 525–529. MR **33** #7376.

14. R. Miller and D. Turnidge, *Some examples from infinite matrix rings*, Proc. Amer. Math. Soc. **38** (1973), 65–67.

15. Y. Miyashita, *Quasi-projective modules, perfect modules, and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ. Ser. I **19** (1966), 86–110. MR **35** #4254.

16. D. G. Northcott, *Lessons on rings, modules, and multiplicities*, Cambridge Univ. Press, London, 1968. MR **38** #144.

17. E. C. Posner, *Prime rings satisfying a polynomial identity*, Proc. Amer. Math. Soc. **11** (1960), 180–183. MR **22** #2626.

18. F. L. Sandomierski, *Modules over the endomorphism ring over a finitely generated projective module*, Proc. Amer. Math. Soc. **31** (1972), 27–31. MR **44** #5335.

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