PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

TENSOR PRODUCTS AND JOINT NUMERICAL RANGE

A. T. DASH[1]

ABSTRACT. It is shown that the joint numerical range of the tensor product of several operators is the cartesian product of their numerical ranges.

Here and in what follows, $\mathscr{H}$ is a complex Hilbert space and by an operator on $\mathscr{H}$ we mean a bounded linear transformation from $\mathscr{H}$ into itself. We denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. Next we need the following definition in the sequel.

DEFINITION. Let $A=(A_1, \cdots, A_n)$ be any n-tuple of operators on $\mathscr{H}$. Then we define the *joint numerical range* of A to be the set $W(A)$ consisting of all $z=(z_1, \cdots, z_n)$ of C^n (the n-dimensional complex space) such that for some f in $\mathscr{H}$ with $\|f\|=1$ we have for each j, $z_j=\langle A_jf,f\rangle$; that is,

$$W(A) = \{\langle Af,f\rangle = (\langle A_1f,f\rangle, \cdots, \langle A_nf,f\rangle): \|f\| = 1, f\in \mathscr{H}\}.$$

For further details about joint numerical range the reader is referred to [**1**]. See also [**4**].

Let $\mathscr{H}_1, \cdots, \mathscr{H}_n$ be complex Hilbert spaces. Further, let I_j be the identity operator and A_j an arbitrary operator on $\mathscr{H}_j$, $1\leqq j\leqq n$. We consider the tensor product of operators T_j $(1\leqq j\leqq n)$ acting on the tensor product space $\mathscr{H}_1\otimes\cdots\otimes\mathscr{H}_n$ defined by

$$T_1 = A_1 \otimes I_2 \otimes \cdots \otimes I_n,$$
$$T_2 = I_1 \otimes A_2 \otimes I_3 \otimes \cdots \otimes I_n,$$

and

$$T_j = I_1 \otimes \cdots \otimes I_{j-1} \otimes A_j \otimes I_{j+1} \otimes \cdots \otimes I_n,$$

in general. The operators T_j obviously commute. For a good account of tensor products of Hilbert spaces and operators, the reader may consult

Presented to the Society, April 3, 1972; received by the editors July 20, 1972.

AMS (MOS) subject classifications (1970). Primary 47A10, 47D20; Secondary 46M05.

Key words and phrases. Algebra of operators, tensor products, convex sets, joint spectrum, joint numerical range, double commutant, commutative Banach algebra, universal property.

[1] This research was partially supported by NRC Grant A07545.

Dixmier [3]. See also [6]. Let $\mathscr{U}$ be the double commutant of $T_1, \cdots, T_n$; that is, the set of all operators on $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$ which commute with any operator commuting with all of T_j. $\mathscr{U}$ is a commutative Banach algebra. A complex vector $z=(z_1, \cdots, z_n)$ of C^n is in the *joint spectrum* $\sigma(T_1, \cdots, T_n)$ of the operators T_j if and only if for all operators $B_1, \cdots, B_n$ in $\mathscr{U}$

$$\sum_{j=1}^{n} B_j(T_j - z_j) \neq I_1 \otimes \cdots \otimes I_n.$$

For facts about joint spectrum see [1], [2] and [4]. It is known that the joint spectrum of the operators T_j $(1 \leqq j \leqq n)$ is the cartesian product of their spectra; that is [5]

$$\sigma(T_1, \cdots, T_n) = \prod_{j=1}^{n} \sigma(T_j) = \prod_{j=1}^{n} \sigma(A_j).$$

The purpose here is to prove an analogous assertion for joint numerical range which is motivated by the paper of the author and Schechter [5].

Before we state our main result, it may be appropriate to point out that the joint numerical range of an n-tuple of operators is not in general convex. Furthermore, it is not known whether or not the joint numerical range of an n-tuple of commuting operators is convex. Consult [1] and [4]. However, the joint numerical range of the operators T_j is convex which is an immediate consequence of the following theorem.

THEOREM. *The joint numerical range of the operators T_j $(1 \leqq j \leqq n)$ is the cartesian product of their numerical ranges; that is,*

$$W(T_1, \cdots, T_n) = \prod_{j=1}^{n} W(T_j) = \prod_{j=1}^{n} W(A_j).$$

Thus it is convex.

We present here several propositions which lead to the proof of the Theorem. To proceed further, we need the following notions and terminologies.

Let $\mathscr{H}$ and $\mathscr{K}$ be any two complex Hilbert spaces. We denote by $\mathscr{H} \odot \mathscr{K}$ their algebraic tensor product (the set of all finite sums $\sum_{j=1}^{n} f_j \otimes g_j, f_j \in \mathscr{H}, g_j \in \mathscr{K}$) and $\mathscr{H} \otimes \mathscr{K}$ their Hilbert space tensor product. We recall that $\mathscr{H} \otimes \mathscr{K}$ is the Hilbert space completion (that is, it is the completion of $\mathscr{H} \odot \mathscr{K}$ for a scalar product which satisfies $\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle$) of $\mathscr{H} \odot \mathscr{K}$. Consult [3] and [6].

PROPOSITION 1. *Let $\mathscr{H}$ and $\mathscr{K}$ be any two complex Hilbert spaces and let E be a complex vector space. If φ is a bilinear mapping of $\mathscr{H} \times \mathscr{K}$*

into E, then there exists a unique linear mapping $V:\mathscr{H}\odot\mathscr{K}\to E$ such that $\varphi(f,g)=V(f\otimes g)$.

This is often known as the "universal property". See **[6]** *and* **[8]**.

The following proposition is crucial to the proof of the Theorem. This is probably well known. See for instance [**8**, Exercise 39.1, p. 410]. However, we were unable to find the exact reference. We will give a proof here for the sake of completeness, and for the benefit of the reader.

PROPOSITION 2. *Consider the tensor product spaces $\mathscr{H}\otimes\mathscr{K}$ and $\mathscr{K}\otimes\mathscr{H}$. Then there exists a unique bounded linear mapping $U:\mathscr{H}\otimes\mathscr{K}\to\mathscr{K}\otimes\mathscr{H}$ such that $U(f\otimes g)=g\otimes f$. This mapping is unitary; that is, it is bijective, and $\langle Uu, Uv\rangle=\langle u, v\rangle$ for all u, v in $\mathscr{H}\otimes\mathscr{K}$.*

PROOF. We take $E=\mathscr{K}\odot\mathscr{H}$, and define a mapping $\varphi:\mathscr{H}\times\mathscr{K}\to\mathscr{K}\odot\mathscr{H}$ by $\varphi(f,g)=g\otimes f$. Evidently φ is bilinear. Thus by Proposition 1 there exists a unique linear mapping $U:\mathscr{H}\odot\mathscr{K}\to\mathscr{K}\odot\mathscr{H}$ such that $U(f\otimes g)=g\otimes f$. Clearly, U is surjective. Now we show that U preserves scalar products. Let $u=\sum_i f_i\otimes g_i$ and $v=\sum_j f'_j\otimes g'_j$ be any two vectors in $\mathscr{H}\odot\mathscr{K}$. Then

$$\begin{aligned}\langle Uu, Uv\rangle &= \left\langle \sum_i g_i\otimes f_i, \sum_j g'_j\otimes f'_j\right\rangle\\ &= \sum_i\sum_j \langle g_i\otimes f_i, g'_j\otimes f'_j\rangle = \sum_i\sum_j \langle g_i, g'_j\rangle\langle f_i, f'_j\rangle\\ &= \sum_i\sum_j \langle f_i, f'_j\rangle\langle g_i, g'_j\rangle = \sum_i\sum_j \langle f_i\otimes g_i, f'_j\otimes g'_j\rangle\\ &= \left\langle \sum_i f_i\otimes g_i, \sum_j f'_j\otimes g'_j\right\rangle\\ &= \langle u, v\rangle.\end{aligned}$$

Since $\mathscr{H}\odot\mathscr{K}$ and $\mathscr{K}\odot\mathscr{H}$ are respectively dense in $\mathscr{H}\otimes\mathscr{K}$ and $\mathscr{K}\otimes\mathscr{H}$, there is a unique continuous linear extension which we also denote by $U:\mathscr{H}\otimes\mathscr{K}\to\mathscr{K}\otimes\mathscr{H}$. Evidently U still preserves scalar products, and hence norm. This implies that its range is complete, and therefore closed. But the range contains $\mathscr{K}\odot\mathscr{H}$, and so must be dense in $\mathscr{K}\otimes\mathscr{H}$. Thus we have $U(\mathscr{H}\otimes\mathscr{K})=\mathscr{K}\otimes\mathscr{H}$.

PROPOSITION 3. *Let A be in $\mathscr{L}(\mathscr{H})$ and B be in $\mathscr{L}(\mathscr{K})$, and U be as given in the preceding proposition. Then:*

(a) *$A\otimes B=U^*(B\otimes A)U$, where star represents adjoint.*

(b) *The numerical range of $A\otimes B$ is the same as that of $B\otimes A$; that is, $W(A\otimes B)=W(B\otimes A)$.*

PROOF. (a). We have

$$\begin{aligned} U^*(B \otimes A)U(f \otimes g) &= U^*(B \otimes A)(g \otimes f) \\ &= U^*(Bg \otimes Af) = Af \otimes Bg \\ &= (A \otimes B)(f \otimes g). \end{aligned}$$

They agree on a dense subset, and therefore the result follows. (b). This is an immediate consequence of (a) and the property of U.

PROPOSITION 4. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ be complex Hilbert spaces, and let A_1 be in $\mathscr{L}(\mathscr{H}_1)$. Then $W(A_1 \otimes I_2) = W(I_3 \otimes A_1) = W(A_1)$.*

PROOF. First we prove that $W(A_1 \otimes I_2) = W(A_1)$. If z is in $W(A_1)$, there exists f in $\mathscr{H}_1$ with $\|f\| = 1$ such that $z = \langle A_1 f, f\rangle$. Let g be any unit vector in $\mathscr{H}_2$. Set $u = f \otimes g$. Clearly, u is in $\mathscr{H}_1 \otimes \mathscr{H}_2$ with $\|u\| = 1$, and

$$\langle (A_1 \otimes I_2)u, u\rangle = \langle A_1 f \otimes g, f \otimes g\rangle = \langle A_1 f, f\rangle = z.$$

This implies that $W(A_1) \subset W(A_1 \otimes I_2)$.

To prove the reverse inclusion, we need the following fact. Let X be a convex subset of C. If $\{z_n\}$ is a sequence of elements in X and $\{\alpha_n\}$ is a sequence of numbers such that $\alpha_n > 0$ and $\sum \alpha_n = 1$, then $\sum_{n=1}^{\infty} \alpha_n z_n$ is in X. This was recently proved by J. P. Williams. Now let z be any element of $W(A_1 \otimes I_2)$. Then there is a unit vector u in $\mathscr{H}_1 \otimes \mathscr{H}_2$ such that $z = \langle (A_1 \otimes I_2)u, u\rangle$. Next we must show that z is in $W(A_1)$. If $\{e_\alpha\}_{\alpha\in\Lambda}$ is an orthonormal basis for $\mathscr{H}_2$, then u can be expressed uniquely as $u = \sum_{\alpha\in\Lambda} f_\alpha \otimes e_\alpha$ for some family vectors $\{f_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{H}_1$ such that $\|u\|^2 = 1 = \sum_{\alpha\in\Lambda} \|f_\alpha\|^2$. Consult [3, p. 22]. Thus we have

$$\begin{aligned} z &= \langle (A_1 \otimes I_2)u, u\rangle \\ &= \left\langle \sum A_1 f_\alpha \otimes e_\alpha, \sum f_\alpha \otimes e_\alpha \right\rangle \\ &= \sum_{\alpha\in\Lambda} \langle A_1 f_\alpha, f_\alpha\rangle. \end{aligned}$$

Now we consider only the nonzero vectors f_α. Clearly from the definition of direct sum, we have that only a countable number of them are nonzero. Thus renumbering them for convenience, we have

$$\begin{aligned} z = \langle (A_1 \otimes I_2)u, u\rangle &= \sum_{\alpha\in\Lambda} \langle A_1 f_\alpha, f_\alpha\rangle \\ &= \sum_{n=1}^{\infty} \langle A_1 f_n, f_n\rangle = \sum_{n=1}^{\infty} \|f_n\|^2 \frac{\langle A_1 f_n, f_n\rangle}{\|f_n\|^2} \\ &= \sum_{n=1}^{\infty} \|f_n\|^2 \langle A_1 \psi_n, \psi_n\rangle, \end{aligned}$$

where $\psi_n = f_n/\|f_n\|$ with $\|\psi_n\|=1$. But $\langle A_1\psi_n, \psi_n\rangle$ is in $W(A_1)$ for each n, and $\sum_{n=1}^{\infty} \|f_n\|^2=1$. Therefore, z is in $W(A_1)$ by the above mentioned theorem of Williams. Thus this proves that $W(A_1\otimes I_2)=W(A_1)$.

To complete the proof of the proposition, it is enough to show that $W(I_3\otimes A_1)=W(A_1)$. This follows readily from Proposition 3 and the proof given above.

In passing, we make the following remark. It is well known that the numerical range of a finite direct sum of operators is the convex hull of the numerical ranges of its summands [7, p. 113]. The proof of the generalization of this assertion to an infinite direct sum of operators follows easily from the theorem of Williams and the techniques used in the preceding proposition. Recall that $A_1\otimes I_2$ could alternatively be regarded as direct sum.

The following lemma is important in sequel. To proceed further, we refer the reader to the introduction for the definition of the operators T_j.

LEMMA. *The numerical range of the operator T_j is the same as that of A_j for each j, $1\leqq j\leqq n$; that is,*

$$W(T_j) = W(A_j), \quad \textit{for all } j,\ 1 \leqq j \leqq n.$$

PROOF. This follows readily from the repeated applications of Proposition 4 and the associative property of the tensor products.

PROOF OF THEOREM. Clearly $W(T_1, \cdots, T_n)\subset\prod_{j=1}^n W(T_j)=\prod_{j=1}^n W(A_j)$. Consult the previous lemma.

Conversely, let $z=(z_1, \cdots, z_n)$ be an element of $\prod_{j=1}^n W(A_j)$. Then there exists f_j in $\mathscr{H}_j$ with $\|f_j\|=1$ such that $z_j=\langle A_jf_j, f_j\rangle$ for all j, $1\leqq j\leqq n$. Then set $u=f_1\otimes\cdots\otimes f_n$. Thus u is a unit vector in $\mathscr{H}_1\otimes\cdots\otimes\mathscr{H}_n$, and $z_j=\langle T_ju, u\rangle=\langle A_jf_j, f_j\rangle$, for all j, $1\leqq j\leqq n$. This implies that z is in $W(T_1, \cdots, T_n)$. Thus $\prod_{j=1}^n W(A_j)\subset W(T_1, \cdots, T_n)$. This proves the theorem.

It is not known in general whether the joint spectrum of a commuting n-tuple of operators is contained in the closure of their joint numerical range [**1**]. However, by the remarks in the introduction we have the following

COROLLARY. $\sigma(T_1, \cdots, T_n)\subset \overline{W(T_1, \cdots, T_n)}$.

PROOF.

$$\begin{aligned}\sigma(T_1, \cdots, T_n) &= \prod_{j=1}^n \sigma(A_j) \subseteq \overline{\prod_{j=1}^n \overline{W(A_j)}} \\ &= \prod_{j=1}^n W(A_j) = \overline{W(T_1, \cdots, T_n)}.\end{aligned}$$

This also follows from above theorem and the fact that the joint spectrum of an n-tuple of commuting operators is contained in the closed convex hull of its joint numerical range [**4**].

ACKNOWLEDGEMENT. I am grateful to Professor J. P. Williams for many helpful and informative conversations.

REFERENCES

1. A. T. Dash, *Joint spectra, joint spectral sets and joint numerical range*, Ph.D. Thesis, University of Toronto, 1969.

2. ———, *Joint spectra*, Studia Math. (to appear).

3. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.

4. N. Dekker, Ph.D. Thesis, Amsterdam, 1969.

5. A. T. Dash and M. Schechter, *Tensor products and joint spectra*, Israel J. Math. **8** (1970), 191–193. MR **41** #5992.

6. A. Guichardet, *Tensor products of C*-algebras*, Lecture Notes Series, nos. 12, 13, Aarhus Universitet, Mathematisk Institut, Denmark, 1969.

7. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

8. F. TREVES, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967. MR **37** #726.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GUELPH, GUELPH, ONTARIO, CANADA (Current address)