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ON THE GROUPS OF INERTIA OF SMOOTH MANIFOLDS[1]

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ABSTRACT. In this paper we study sufficient conditions for a manifold M^n to have $I(M^n)=\{0\}$. We also prove that if M^n is a smooth manifold of dimension n, $n\equiv 2 \pmod 8$, with $w_2(M^n)\neq 0$, then $I(M^n)\neq 0$.

Introduction. In [**8**] Kosinski defined, for a smooth oriented closed manifold M^n, of $I(M^n)$, to be the subgroup of θ^n consisting of all homotopy spheres Σ^n for which there exists an orientation preserving diffeomorphism from M^n to $M^n \# \Sigma^n$. θ^n is the group of homotopy spheres defined by Kervaire and Milnor in [**6**]. $\#$ stands for connected sum of two manifolds. $I(M^n)$ is the group of inertia of M^n.

In §1, we give some sufficient conditions for a manifold M^n to have $I(M^n)=\{0\}$. In §2, we study smooth manifolds of dimension n, $n\equiv 2$ (mod 8) and with $\omega_2(M)\neq 0$ and show that $I(M^n)$ for these manifolds does not vanish. All manifolds considered in this paper are oriented C^∞ manifolds, unless otherwise stated.

1. **Manifolds with zero groups of inertia.**

In this section we give some sufficient conditions for a manifold M^n in order to have $I(M^n)=0$.

DEFINITION. We say that a manifold M^n satisfies property (S) if $H_i(M^n, Z)$ is a cyclic group for all $i\geqq 0$.

THEOREM. *Let M^n be a smooth closed manifold of dimension $n\geqq 7$ satisfying property* (S). *Then if M^n imbeds smoothly in S^{n+1} and bounds a simply connected region in S^{n+1}, then $I(M^n)=0$.*

PROOF. Let $M^n\subseteq S^{n+1}$. M^n separates S^{n+1} into two components A, B such that

$$A \cup B = S^{n+1}; \qquad A \cap B = M^n = bA = bB.$$

A, B are smooth manifolds with boundary M^n.

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Let Σ^n be a homotopy sphere that represents an element in $I(M^n)$, then there exists an orientation preserving differomorphism

$$\phi: M^n \# \Sigma^n \to M^n.$$

Let $B'=B\#(\Sigma^n\times I)$ be the connected sum of B and $\Sigma^n\times I$ along bB and $\Sigma^n\times(0)$. Let W^{n+1} be the manifold obtained from A and $B'=B\#(\Sigma^n\times I)$ by pasting M^n and $M^n\#\Sigma^n$ via the diffeomorphism ϕ. $bW=\Sigma^n$. It follows from van Kampen's theorem that W is simply connected.

We consider the Mayer-Vietoris homology sequences with integer coefficients for S^{n+1} and for W^{n+1}. For $i+1<n+1$,

$$0 = H_{i+1}(S^{n+1}) \to H_i(M^n) \xrightarrow{i_A\oplus(-i_B)} H_i(A)\oplus H_i(B) \to H_i(S^{n+1})$$
$$\to H_{i+1}(W) \to H_i(M^n) \xrightarrow{i_A\oplus(-i_B\circ\phi^{-1})} H_i(A)\oplus H_i(B') \to H_i(W^{n+1}) \to.$$

We have

$$H_i(M^n) \xrightarrow{i_A\oplus(-i_B)} H_i(A)\oplus H_i(B)$$

is an isomorphism and $\ker i_A \cap \ker i_B=0$.

Since $H_i(M^n)$ is cyclic, we have both $\ker i_A$ and $\ker i_B$ are cyclic groups of different orders. On the other hand we can identify the homology groups of M^n with those of $M^n\#\Sigma^n$ and consider ϕ_*^{-1} as an automorphism of $H_i(M^n)$ onto $H_i(M^n\#\Sigma^n)$ and this automorphism leaves $\ker i_B$ invariant. Therefore $\ker i_A \cap \ker(i_{B'}\circ\phi_*^{-1})=0$; and hence $i_A+(-i_{B'}\circ\phi_*^{-1})$ is injective. We show $i_A+(-i_{B'}\circ\phi_*^{-1})$ is surjective. Let $x\in H_i(A)\oplus H_i(B')$. x corresponds to an element, call it x again, in $H_i(A)\oplus H_i(B)$. Hence $x=a+b$, $a\in H_i(A)$, $b\in H_i(B)$. Put $y=i_A^{-1}(a)+(-i_B^{-1}b)$; $y'=i_A^{-1}(a)+(-\phi_* i_B^{-1}b)$. Hence $[i_A+(-i_{B'}\circ\phi_*^{-1})](y')=x$. Thus $i_A+(-i_{B'}\circ\phi_*^{-1})$ is an isomorphism for all $i<n$.

Therefore, $H_i(W)=0$, $1\leq i\leq n$; and since $\Pi_1 W=0$, W is a contractible manifold and hence $\Sigma^n=bW$ is diffeomorphic to S^n, i.e. it represents the zero element in θ^n [6]. This proves that $I(M^n)=0$.

COROLLARY. *Let M^n be a smooth manifold satisfying property* (S). *Let T be the total space of the normal sphere bundle of M^n in S^{n+k} for $k>n+1$. Then $I(T)=0$.*

As an example of this situation, we have:

COROLLARY. *Let T be the total space of the normal sphere bundle of $P_n(c)$, the complex projective space of dimension n, in S^{n+k}, $k>2n+1$, then $I(T)=0$.*

In [5] Steer and Brown proved that $I(V_n)\neq 0$, where V_n is the Stiefel manifold of unit tangent vectors to S^n, n odd, $n\neq 1, 3, 7$. In fact $I(V_n)\supseteq \theta^{2n-1}(\partial\pi)=$the subgroup of θ^{2n-1} consisting of those homotopy spheres that bound parallelizable manifolds. Since S^7 is parallelizable, $V_7 \approx S^7 \times S^6$ and by the Theorem, $I(V_7)=0$. We show next that this is a special case of a more general situation, namely:

COROLLARY. *Let M^n be a simply connected manifold satisfying property* (S), *and M is parallelizable, then if T^{2n-1} is the total space of the tangent sphere bundle of M^n*, $(T^{2n-1} \approx M^n \times S^{n-1})$, $I(T^{2n-1})=0$.

PROOF. It is enough to show T imbeds in S^{2n} since T satisfies (S). Imbed M^n in S^{2n}. The obstruction to the triviality of the normal bundle of $M^n \subset S^{2n}$ lies in $\ker J_n \cap \ker i_n$ where

$$J_n : \Pi_{n-1}(SO_n) \to \Pi_{2n-1}(S^n)$$

is the J-homomorphism, and

$$i_n : \Pi_{n-1}(SO_n) \to \Pi_{n-1}(SO)$$

is the map induced by the inclusion of SO_n in SO. But $\ker J_n \cap \ker i_n = 0$, which implies that the normal bundle of $M^n \subset S^{2n}$ is trivial, and hence the tangent sphere bundle which is diffeomorphic to $M^n \times S^{n-1}$ imbeds in S^{2n}.

We recall that the mod 2 semicharacteristic of an odd dimensional manifold M^{2n-1} is defined to be $\sum_{i=0}^{n-1} b_i$, where b_i is the rank of the mod 2 homology group $H_i(M, Z_2)$.

THEOREM (ADAMS-KERVAIRE). *Let M be a closed Π-manifold.*

(i) *When M is even dimensional, M is parallelizable if and only if its Euler characteristic is zero.*

(ii) *When M is odd dimensional, M is parallelizable if and only if either its dimension is* 1, 3, 7, *or its mod* 2 *semicharacteristic is even.*

For a proof, see [7].

Using this theorem, we get

COROLLARY. *Let M^n be a simply connected Π-manifold satisfying property* (S). *If n is even, assume the Euler characteristic is zero. If n is odd, $n \neq 1, 3, 7$, assume the* mod 2 *semicharacteristic is zero. Then the group of inertia of the tangent sphere bundle of M^n is zero.*

COROLLARY. *The group of inertia of the tangent sphere bundle of V_n is zero.*

2. **Manifolds with nonzero groups of inertia.** In this section we give some sufficient conditions on a closed connected smooth manifold in order to have the group of inertia contain more than one element. Before stating the main theorem we recall some well-known facts.

Let ξ be a universal spin $(8k)$ bundle, let $D(\xi)$, $S(\xi)$ be the total spaces of the associated disk and sphere bundles respectively. Put

$$M(\text{spin } 8k) = D(\xi)/S(\xi)$$

for the universal Thom space of ξ. It is known that

$$\Pi_{n+8k}(M \text{ spin } 8k) \cong \Omega_n^{\text{spin}}$$

where Ω_n^{spin} is the nth spin cobordism group. Moreover, for $n \not\equiv 0 \pmod 4$, Ω_n^{spin} is finite. Let

$$P_n : \theta^n \to \Omega_n^{\text{spin}}$$

be the map which assigns to each element $\Sigma^n \in \theta^n$ its cobordism class in Ω_n^{spin}. Let $K_n = \text{kernel } P_n$.

The proof of the following proposition is in [**2**]; see also [**9**].

PROPOSITION. *For* $r \equiv 1, 2 \bmod 8$, $r > 2$, *the image of the homomorphism* $P_r : \theta^r \to \Omega_r^{\text{spin}}$ *is nontrivial.*

Now we state the main theorem of this section.

THEOREM. *Let* M^n *be a closed* 1*-connected smooth manifold*, $n \equiv 2 \pmod 8$, $n > 2$. *Then* $I(M^n) \subseteq K_n$ *if and only if* $\omega_2(M^n) = 0$, *the second Whitney class of* M^n. *In particular if* $\omega_2(M^n) \neq 0$, $I(M^n) \neq 0$.

For the proof of the "if" half see [**4**] or [**9**]. Next we recall the theorem of J. F. Adams [**1**, Theorem 1.2].

THEOREM (ADAMS). *Suppose that* $r \equiv 1$ *or* $2 \bmod 8$ *and* $r > 0$. *Then* $\Pi_r^s = \lim_{n\to\infty} \Pi_{n+r}(S^n)$ *contains an element* μ_r *of order* 2 *such that if* $i_* : \Pi_{r+8k}(S^k) \to \Pi_{r+8k}(M \text{ spin } 8k)$ *is the homomorphism induced by inclusion, then* $i_*\mu_r \neq 0$.

We now proceed to prove that the homotopy spheres represented by the μ_n lie in $I(M^n)$ for every 1-connected manifold with $\omega_2(M^n) \neq 0$, $n \equiv 2 \bmod 8$. We start by recalling some results of Novikov [**10**].

Let M^n be a closed manifold; $T(\nu_M)$ be the Thom space of the normal bundle of $M^n \subset S^{n+N}$ for N large. There exists a natural map

$$k : S^N \to T(\nu_M)$$

induced by an inclusion of a fiber.

Let M_1^n be a manifold tangentially equivalent to M^n. Put $B(M_1^n) =$ the set of all homotopy classes $\alpha \in \Pi_{n+N}(T)$ for which there exists a representative f_α

$$f_\alpha : S^{n+N} \to T(\nu_M)$$

such that f_α is transverse regular to $M \subset T(\nu_M)$, $f_\alpha^{-1}(M) = M_1$ and $f_\alpha | M_1$ is the tangential equivalence between M_1 and M.

For any homotopy sphere Σ^n, Kervaire and Milnor in **[6]** defined $P(\Sigma^n)\subseteq\Pi_n^s$ to be the set of all homotopy classes in Π_n^s which can be obtained from all possible framings of $\Sigma^n\subseteq S^{n+N}$, using the Thom-Pontriagin construction. Taking $M=M_1$, we have

LEMMA (NOVIKOV) **[10]**. *$B(M^n\#\Sigma^n)\supseteq B(M^n)+k_*P(\Sigma^n)$ where k_*: $\Pi_{n+N}(S^N)\to\Pi_{n+N}(T)$ is the map induced from k.*

COROLLARY. *Assume there exists an element $\gamma\in P(\Sigma^n)$ such that $k_*\gamma=0$, then there exists a homotopy sphere $\Sigma_1^n\in\theta^n(\partial\pi)$ such that $\Sigma^n\#\Sigma_1^n\in I(M^n)$.*

PROOF. Let $\alpha\in B(M^n)$ be a "normal invariant" of M^n. It follows from the assumption and the last lemma that α is also a normal invariant of $M^n\#\Sigma^n$. Now by a theorem of Browder-Novikov, if two manifolds correspond to the same normal invariant, they are diffeomorphic modulo $\theta^n(\partial\pi)$ **[10]**.

LEMMA. *Let M^n be a 1-connected closed manifold, $n\equiv 2 \bmod 8$, $n>2$. Assume $\omega_2(M^n)\neq 0$. Let $T=T(\nu_M)$, the Thom space of the normal bundle of M in S^{n+N}. Then $k_*\mu_n=0$, where the μ_n are Adams elements.*

PROOF. Since $\Pi_1 M=\{1\}$, and $\omega_2(M^n)\neq 0$, there exists a sphere S^2 imbedded in M^n with nontrivial normal bundle (in fact with a characteristic class α, the nonzero element of $\Pi_1(SO_{n-2})=Z_2$).

The normal bundle ν of $M^n\subset S^{n+N}$, when restricted to S^2, is just $E^{N-n+2}\alpha=$ the iterated suspension of α. Put $\beta=E^{N+n+2}\alpha\in\Pi_1(SO_N)$. If T_0 is the Thom space of β, we have $T_0\subseteq T$. But, by the definition of the J-homomorphism we have

$$T_0 = S^N \cup_{J\beta} D^{N+2}, \qquad J\beta\in\Pi_{N+1}(S^N) = \Pi_1^s.$$

Let $f: S^{N+n}\to S^N$ be a representative of μ_n. Adams defined $\mu_n=\mu_{8s+2}$ to be $\eta\mu_{8s+1}$ where η is the generator of Π_1^s.

Therefore we have the following diagram.

$$\begin{tikzcd}
S^{N+n} \arrow[rr, "f"] \arrow[dr, "\mu_{n-1}"'] & & S^N \arrow[r, "i"] & S^N \cup_{J\beta} D^{N+2} \arrow[r, "j"] & T \\
& S^{N+1} \arrow[ur, "\eta"'] & & &
\end{tikzcd}$$

Since $J\beta=\eta$, $i\circ\eta\simeq *$, and $j\circ i\circ f\simeq *$. Thus, $k_*\mu_n=0$.

REMARK. The lemma remains true if $\Pi_1 M\neq(1)$ and $\Pi_2(M)\neq 0$ but $H_2\Pi_1$, the second homology group of Π_1, vanishes. This is enough to guarantee the existence of an imbedded $S^2 \subset M$ in each homology class of H_2M.

Proof of the "only if" part of the main theorem. Let Σ^n be a homotopy sphere represented by μ_n, by the Thom-Pontriagin construction. The results of [2] imply that Σ^n does not bound a spin manifold (see also [9]) and $\Sigma^n \notin k_n$. By the last lemma and previous corollary there exists $\Sigma_1^n \in \theta^n(\partial\pi)$ so that $\Sigma^n \# \Sigma_1^n \in I(M^n)$. But for n even, $\theta^n(\partial\pi)=0$ [6]; and so $\Sigma^n \in I(M^n)$.

This completes the proof of the theorem.

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