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ON RETRACEABLE SETS WITH RAPID GROWTH

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ABSTRACT. We combine a refinement of a recent theorem of A. N. Degtev with a result of our own, in order to derive a general theorem about regressive sets which has the following

COROLLARY. *If A is any point-decomposable π_1^0 set then A has an infinite π_1^0 subset B such that B has "highly" dense-simple complement and, moreover, all infinite π_1^0 subsets of B are effectively decomposable in a strong sense (namely, they are all retraceable).*

1. **Introduction and principal theorem.** Various recent articles ([1], [3], [4], [10], and, implicitly, [9]) have dealt with (infinite) retraceable sets A having the following property: if p_A is the principal function of A (i.e., the function which enumerates A in order of magnitude) and if φ_e is any partial recursive function, then $\varphi_e(p_A(n))<p_A(n+1)$ holds for almost all n. Let us refer to this phenomenon as *property* (P), independently of whether A is retraceable. In [10], we proved some general theorems about regressing functions which immediately imply the following result:

THEOREM 1 (CF. [10, THEOREMS 3I.1 AND 3I.2]). *If A is any infinite regressive set of natural numbers, then there exist sets B and C such that C is r.e., $B=A\cap\bar{C}$, and B is a retraceable set having both property* (P) *and, also, the property (which we shall call property* (Q)) *that $p_B(n)>\varphi_e(n)$ holds for all sufficiently large n, for any partial recursive function φ_e. (As is noted in* [10], *it is in fact the case that* (P)$\Rightarrow$(Q) *holds for all retraceable sets.)*

Naturally, we refer in both (P) and (Q) only to those x for which $\varphi_e(x)$ is defined.

(For background information on retraceable and regressive sets, the reader may consult [2].)

For the convenience of the reader, we shall briefly (and, in the case of Theorem 3I.1, very informally) indicate the content of Theorems 3I.1

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and 3I.2 of [**10**], from which it will be clear how they combine to yield Theorem 1 above. Theorem 3I.1 [**10**] is a rather technical result which, very roughly speaking, asserts that if f is any partial recursive regressing function, then there is an r.e. set C and a partial recursive retracing function p, such that if A is (the set of nodes of) an infinite branch of the graph of f, then $\bar{C} \cap A$ is an infinite retraceable set retraced by p and, moreover, $\bar{C} \cap A$ when arranged in natural order has very strong order-preservation properties with respect to the class of partial recursive functions. (Actually, Theorem 3I.1 of [**10**] asserts more; we have indicated only the portion we need.) Theorem 3I.2 [**10**] asserts that if the infinite branches of a partial recursive retracing function have the order-preservation properties of [**10**, Theorem 3I.1], then they are all "thin" in the sense of enjoying *both* property (P) and property (Q). Theorem 1 of the present paper follows, since to be regressive is precisely to be (the set of nodes of) a branch of a partial recursive regressing function.

In his interesting recent paper [**1**], A. N. Degtev has proved the following theorem (among others):

THEOREM 2 (DEGTEV). *Suppose A is an infinite retraceable set such that $\bar{A}$ is r.e., and such that A has property* (P). *Then if B is any infinite co-r.e. subset of A, B is retraceable.*

We here observe that a somewhat stronger form of Theorem 2 holds, namely:

THEOREM 2′. *If A is any infinite retraceable set having property* (P), *and if B is any co-r.e. set such that $A \cap B$ is infinite, then $A \cap B$ is retraceable.*

Though the proof of Theorem 2′ is not difficult, the theorem itself was overlooked by the author of the present note in his fairly extensive investigation [**10**] of sets with property (P). As an adequate indication of the proof, we offer the following: Since A has property (P), the principal function p_A of A satisfies the condition

$$(\exists m)(\forall n)[(n > m \,\&\, g(p_A(n)) \text{ defined}) \Rightarrow p_A(n+1) > g(p_A(n))]$$

where $g(n) \simeq_{df} (\mu y)$ [y is the Gödel number of a computation of $\varphi_e(n)$] with e chosen so that $\bar{B}$=the domain of φ_e. This fact allows us to tell of a number $p_A(n+1)$ whether $p_A(n)$ is in $\bar{B}$, with finitely many exceptions (which of course do not matter).[2]

We come now to our main assertion and its corollary. In the statement

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[2] It is easily seen from this proof that a further strengthening of Theorem 2 is possible; namely, in Theorem 2′ we need not require that B be co-r.e. but only that it lie in the boolean algebra generated by the r.e. sets.

of the corollary, *highly dense-simple* means r.e. with complement having property (Q); while *point-decomposability* is to be understood as defined in **[8]**. (The notion of (not necessarily high) *dense simplicity* was first introduced in [7].)

THEOREM 3. *Let A be an infinite regressive set. Then there exists a recursively enumerable set B such that*

(i) *$A \cap \bar{B}$ is infinite and retraceable and has properties* (P) *and* (Q),

(ii) *$(\forall C)[(C$ r.e.$\&B\subseteq C\&A\cap\bar{C}$ infinite$)\Rightarrow A\cap\bar{C}$ is retraceable$]$.*

PROOF. Applying Theorem 1 to A, we obtain an r.e. set B such that $A\cap\bar{B}$ is infinite, retraceable, and has properties (P) and (Q). By property (P) and Theorem 2′, $A\cap\bar{C}$ is retraceable for any r.e. set C satisfying $B\subseteq C\&A\cap\bar{C}$ infinite, and we are done.

REMARK. It is easily shown that property (P) is hereditary for retraceable subsets; hence, in the statement of Theorem 3, we can strengthen (ii) by asserting that $A\cap\bar{C}$ has property (P) as well as being retraceable.

As remarked in **[1]**, if A is r.e. and coinfinite and can be extended to an r.e. superset B such that B has a point-decomposable complement, then A can be extended to an r.e. set C such that $\bar{C}$ is infinite, immune, and regressive. We therefore obtain the following corollary to Theorem 3, which provides yet another refinement (see Theorems in [7], **[6]**, and **[11]**) of Martin's result that hypersimple sets need not have maximal supersets:

COROLLARY 1. *Let A be any r.e. set which can be extended to an r.e. set B having a point-decomposable complement. Then A can be extended to a highly dense-simple set C all of whose co-infinite r.e. extensions are co-retraceable.*

PROOF. Property (Q), for the complement of an r.e. set, is precisely our notion of *high* dense simplicity.

REMARK. A weaker version of Corollary 1 is certainly already present in **[1]**, since Degtev there exhibits his own construction of a particular co-r.e. retraceable set having property (P). The latter construction can in fact be modified to take place *inside* a given infinite retraceable set with r.e. complement, although this is not done in **[1]**; such a modification leads at once to another proof of Theorem 1 for the special case in which $\bar{A}$ is r.e.

2. **A further application of Theorem 1, and a concluding remark relating [1] and [5].** C. G. Jockusch has proved that no r.e. set can be both dense simple (in the sense of [7]) and strongly effectively simple. (See **[5]** for the meaning of *strong effective simplicity*; the standard example is the original simple-but-not-hypersimple set constructed by E. L. Post.)

From Jockusch's result and Theorem 1, since strong effective immunity is trivially hereditary, we have:

COROLLARY 2. *The complement of an infinite, co-r.e. regressive set cannot be strongly effectively simple.* (*It is not hard to show, on the other hand, that such a set can be merely effectively simple*; *again, see* **[5]** *for the definition of effective simplicity. We are indebted to Jockusch for pointing out Corollary* 2.)

PROOF. By Theorem 1 we have that each infinite, co-r.e. regressive set can be shrunk to an infinite, co-r.e. retraceable set having property (Q) and hence having a highly dense-simple complement. Now apply Jockusch's theorem on the incompatibility of dense simplicity and strong effective simplicity, noticing that any highly dense-simple set is certainly dense simple in the sense of [7].

Jockusch has suggested that we remark also on the fact that Degtev has shown, in **[1]**, that every semirecursive regressive set is either r.e. or co-r.e. This not only answers a question raised in **[5]**, but, in light of Corollary 2 above, it shows that Theorem 6.4 of **[5]** is vacuous.

We would like to emphasize, in conclusion, that the really crucial observation for this note is Degtev's simple but rather striking Theorem 2.

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