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CRITERIA FOR COMPACTNESS AND FOR DISCRETENESS OF LOCALLY COMPACT AMENABLE GROUPS

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ABSTRACT. Let G be a locally compact group $P(G)=\{0\leqq\phi\in L_1(G); \int\phi(x)\,dx=1\}$ and $(l_a f)(x) = {}_a f(x)=f(ax)$ for all $a, x\in G$ and $f\in L^\infty(G)$. $0\leqq\Psi\in L^\infty(G)^*$, $\Psi(1)=1$ is said to be a [topological] left invariant mean ([TLIM] LIM) if $\Psi({}_a f)=\Psi(f)$ $[\Psi(\phi * f)=\Psi(f)]$ for all $a\in G$, $\phi\in P(G), f\in L^\infty(G)$. The main result of this paper is the

THEOREM. *Let G be a locally compact group, amenable as a discrete group. If G contains an open σ-compact normal subgroup, then LIM=TLIM if and only if G is discrete. In particular if G is an infinite compact amenable as discrete group then there exists some $\Psi\in$ LIM which is different from normalized Haar measure. A harmonic analysis type interpretation of this and related results are given at the end of this paper.*[2]

Introduction. It was known to Fred Greenleaf that if T is the circle group then there are at least two different linear translation invariant functionals $\Psi\geqq 0$ on $L^\infty(T)$ with $\Psi(1)=1$. One of them is certainly that given by the normalized Haar measure λ on T.

It is easy to show and it is known that on any compact G, λ is the unique $0\leqq\Psi\in L^\infty(G)^*$, $\Psi 1=1$ which satisfies the stronger invariance property $\Psi(\phi * f)=\Psi(f)$ for all $f\in L^\infty(G)$, $\phi\in P(G)$ (i.e. λ is the unique TLIM on $L^\infty(G)$). This is the case since $\phi * f\in C(G)$ for all $\phi\in P(G)$, $f\in L^\infty(G)$ and if $\Psi\in$ TLIM then $\Psi\in$ LIM [6, p. 25]. Thus $\Psi=\lambda$ at least on $C(G)$. But then for all $f\in L^\infty(G), \Psi(f)=\Psi(\phi * f)=\lambda(\phi * f)=\lambda(f)$.

It seemed to Greenleaf that for any *compact* infinite G, which is *amenable as a discrete* group, there exist at least two different LIM's on $L^\infty(G)$. Our main result in this paper implies the

THEOREM. *Let G be a locally compact group which is abelian or σ-compact and amenable as a discrete group. Then LIM=TLIM if and*

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Thanks are due to the referee for a simplification in the proof of Proposition 1.

[2] The main result of this paper has independently been obtained by W. Rudin in a recent paper *Invariant means on L^∞*, Studia Math. **44** (1972), 219–227, which was not in print when our paper was sent for publication. (See Addition at the end of present paper.)

only if G is discrete. In particular on any compact infinite G which is amenable as discrete there exists some $\Psi \in LIM$ different from the normalized Haar measure.

Let H $[H_c]$ be the linear span of $\{f - l_a f; f \in L^\infty(G), a \in G\}$ $[\{f - \phi * f; \phi \in P(G), f \in L^\infty(G)\}]$ and for $A \subset L^\infty(G)$ let $\bar{A}$ $[\bar{A}^*]$ denote the norm $[w^*]$ closure of A in $L^\infty(G)$. In any locally compact group one has $\bar{H} \subset \bar{H}_c \subset \bar{H}_c^* = \bar{H}^* \subset L^\infty(G)$. Our last result (combined with some known facts) when restricted to σ-compact locally compact abelian groups runs as follows:

PROPOSITION. (i) *If G is compact and infinite then $\bar{H} \subsetneqq \bar{H}_c = \bar{H}_c^* = \bar{H}^* = \{f \in L^\infty(G); \lambda f = 0\}$.*

(ii) *If G is not compact then $\bar{H} \subset \bar{H}_c \subset \bar{H}_c^* = \bar{H}^* = L^\infty(G)$. Moreover $L^\infty(G)/\bar{H}_c$ is a nonseparable Banach space and $\bar{H} = \bar{H}_c$ iff G is discrete.*

We conjecture at the end that for *any* locally compact amenable group G, if G is noncompact then $L^\infty(G)/\bar{H}_c$ is a nonseparable Banach space and if G is nondiscrete then $\bar{H}_c/\bar{H}$ is nonseparable (with induced quotient norms).

Some more notations. Unless otherwise specified we assume the notations and definitions of Hewitt-Ross [7].

If G is a locally compact group λ will denote a fixed left Haar measure (with $\lambda(G)=1$ if G is compact), we write sometimes $\int \phi(x)\,dx$ instead of $\int \phi\, d\lambda$.

$\Psi \in L^\infty(G)^*$ is said to be [topologically] left invariant if $\Psi(l_a f) = \Psi(f)$ $[\Psi(\phi * f) = \Psi(f)]$ for all $f \in L^\infty(G)$, $\phi \in P(G)$, $a \in G$ (where $l_a f(x) = {}_a f(x) = f(ax)$). If Ψ satisfies in addition $\Psi \geqq 0$ and $\Psi(1)=1$ then Ψ is said to be a [topological] left invariant mean ([TLIM] LIM resp.). The set of all [TLIM] LIM is also denoted by [TLIM] LIM. Analogously we define [TRIM] RIM the sets of [topological] right invariant means.

We stress that LIM, TLIM are both included in $L^\infty(G)^*$. The locally compact group G is said to be amenable if LIM $\neq \varnothing$ (or equivalently if TLIM $\neq \varnothing$ see [6]). G is said to be amenable as discrete if G_d (i.e. G with the discrete topology) is amenable.

We write sometimes LIM(G), TLIM(G) to emphasize dependence on the group G. If $A \subset G$, 1_A denotes the function 1 on A and zero otherwise. If $\Psi \in L^\infty(G)^*$, we write $\Psi(B)$ instead of $\Psi(1_B)$ for measurable $B \subset G$. 1 also stands for the constant one function on G.

PROPOSITION 1. *Let G be any noncompact locally compact group and $\phi \in TRIM$. If B is a measurable set and $\lambda(B) < \infty$ then $\phi(B) = 0$.*

PROOF. Let $\phi_\alpha \in P(G)$ be such that $\phi_\alpha \to \phi$ in w^* and let $\eta \in P(G)$ be such that $0 \leq \eta(x) \leq \varepsilon$ for all $x \in G$. Then

$$|\phi_\alpha * \eta(x)| = \left|\int \phi_\alpha(y)\eta(y^{-1}x)\, d\lambda\right| \leq \varepsilon \int \phi_\alpha(y)\, d\lambda = \varepsilon.$$

Furthermore if $f \in L^\infty(G)$ then

$$(\phi_\alpha * \eta)(f) = \phi_\alpha(f * \tilde{\eta}) \to \phi(f * \tilde{\eta}) = \phi(f).$$

(See Wong [10, p. 352].) Hence if $f \in L^\infty \cap L^1$ then $|(\phi_\alpha * \eta)(f)| \leq \int \varepsilon |f|\, d\lambda$ so $|\phi f| \leq (\int |f|\, d\lambda)\varepsilon$. Thus $\phi f = 0$.

We need the following, probably known, proposition for which we were unable to find a reference.

PROPOSITION 2. *Let G be a σ-compact nondiscrete locally compact group. Then for any $\varepsilon > 0$ there exists an open dense set $B \subset G$ with $\lambda(B) < \varepsilon$.*

PROOF. It is enough to show the existence of a dense set $D \subset G$ with $\lambda(D) = 0$ and the regularity of λ would imply that for some open $D \subset B$, $\lambda(B) < \varepsilon$.

If G is separable then there is some countable dense $D \subset G$. Clearly $\lambda(D) = 0$.

Assume now that G is arbitrary and $N \subset G$ a closed normal subgroup. Let $\theta: G \to G/N$ be the canonical map. If $D \subset G$ with θD dense in G/N then DN is dense in G. In fact if $U \subset G$ is open with $U \cap DN = \varnothing$ then $UN \cap DN = \varnothing$ so $\theta^{-1}(\theta U \cap \theta D) = UN \cap DN = \varnothing$ thus $\theta U \cap \theta D = \varnothing$ and θU is open in G/N which cannot be.

If G is σ-compact nondiscrete let $U \subset G$ be an open neighborhood of the identity and let $G_0 = \bigcup_{-\infty}^{\infty} U^n$. Then G_0 is open compactly generated and there are countably many left cosets of G w.r.t. G_0. The left Haar measure of G_0 can be taken to be the restriction to G_0 of the left Haar measure λ on G. It is enough hence to show that there is a dense null set $D \subset G_0$ i.e. we can and shall assume that G is compactly generated nondiscrete. Let then U_n be a sequence of identity neighborhoods in G with $\lambda(U_n) \to 0$ and let $N \subset \bigcap_1^\infty U_n$ be a compact normal subgroup such that G/N is metrizable separable (see [7, p. 71]). (G/N is not discrete since $\lambda N = 0$ so N is not open.) Let $D = \{d_i\}_1^\infty \subset G$ be such that its image in G/N is dense. Then $DN \subset G$ is dense and $\lambda(DN) = 0$ since D is countable.

We need in the sequel the following proposition (not in its full force) which is in part due to Følner [3] for discrete amenable groups.

PROPOSITION 3. *Let G be a locally compact group which is amenable as a discrete group. For $f \in L^\infty(G)$ let $M(f) = \sup\{\phi(f);\ \phi \in LIM\}$. Then* *for all* $f \in L^\infty(G)$

$$Mf = \inf_{\mathscr{A}} \operatorname{ess\,sup}_x \left[\frac{1}{n}\sum_1^n f(a_i x)\right]$$

the inf *being taken over the set* $\mathscr{A}$ *of all finite tuples* $(a_1, \cdots, a_n)$ *of elements of* G.

PROOF. Let H be the linear span of $\{f - l_a f;\ a \in G, f \in L^\infty(G)\}$. It is known (and due to Følner [**3**, p. 6] for discrete G) that:

$$M(f) = \inf_{h \in H} \operatorname{ess\,sup}_x (f(x) + h(x))$$

for all $f \in L^\infty(G)$. (For an easy proof see [**5**, p. 401].)

Also if $\phi \in \mathrm{LIM}$ then $\phi f = \phi(n^{-1} \sum_1^n l_{a_i} f)$ hence

$$Mf \leqq \inf_{\mathscr{A}} \operatorname{ess\,sup}_x \frac{1}{n}\sum_1^n f(a_i x).$$

Let now $\varepsilon > 0$ and $h_0 \in H$ be such that $M(f) + \varepsilon > \operatorname{ess\,sup}_x (f(x) + h_0(x))$. So $M(f) + \varepsilon \geqq f(x) + h_0(x)$ locally a.e. and a fortiori $M(f) + \varepsilon \geqq n^{-1} \sum_1^n l_{a_i}(f(x) + h_0(x))$ loc. a.e. for all $a_1, \cdots, a_n$ in G. We claim that a finite set $\{b_1, \cdots, b_k\} \subset G$ can be chosen such that $|k^{-1} \sum l_{b_i} h_0(x)| < \varepsilon/2$ loc. a.e. This would imply that $M(f) + 3/2\varepsilon \geqq k^{-1} \sum l_{b_i} f(x)$ loc. a.e., i.e. that

$$M(f) \geqq \inf_{\mathscr{A}} \operatorname{ess\,sup}_x \frac{1}{n}\sum_1^n l_{a_i} f(x)$$

which would end this proof.

To prove this claim let $h_0 = \sum_1^n [f_i - l_{c_i} f_i]$. For the finite set $F = \{c_1, \cdots c_n\}$ choose a finite subset $A = \{b_1, \cdots, b_k\}$ to satisfy $c(c_i A \sim A) < \delta c(A)$ for $1 \leqq i \leqq n$ where $c(B)$ stands for the cardinality of B and $\delta = \varepsilon(\max_{1 \leqq i \leqq n} \|f_i\|)^{-1} n^{-1}$. Such A can be found by Følner's characterization of discrete amenable groups [**2**] (see Namioka [**9**, p. 22]). Then for each $i \leqq n$

$$\left|\frac{1}{k}\sum_{j=1}^k l_{b_j}(f_i - l_{c_i} f_i)\right| = \left|-\frac{1}{k}\sum_{j=1}^k (l_{c_i b_j} - l_{b_j}) f_i\right|$$
$$\leqq c(c_i A - A)\,\|f_i\|/c(A) < \delta \|f_i\| \leqq \varepsilon/n.$$

Therefore $|k^{-1} \sum l_{b_i} h_0(x)| \leqq \varepsilon/2$ loc. a.e. which finishes this proof.

REMARKS. 1. It seems that this proposition does not hold true if G is not amenable as a discrete group (even in the case that G is compact).

2. If $m(f) = \inf\{\phi(f);\ \phi \in \mathrm{LIM}\}$ then

$$m(f) = -M(-f) = \sup_{\mathscr{A}} \left[\operatorname{ess\,inf}_x \frac{1}{n}\sum_1^n f(a_i x)\right].$$

3. One can show in a similar way that the support functional of the set of two-sided invariant means is $M_0 f = \inf_{\mathscr{A}} \operatorname{ess\,sup}_x (1/nm) \sum_{i,j} f(a_i x b_j)$ where $\mathscr{A}$ is the set of all pairs of finite tuples $(a_1, \cdots, a_n)(b_1, \cdots, b_m)$ of elements of G.

THEOREM 1. *Let G be a locally compact σ-compact group which is amenable as a discrete group. If $LIM = TLIM$ then G is discrete.*

PROOF. Assume that G is not discrete and let O be an open dense set in G with $\lambda(O) < \frac{1}{2}$. Thus, if G is not compact then $\phi(O) = 0$ for all $\phi \in \mathrm{TRIM}$ hence $\Psi(O^{-1}) = 0$ for all $\Psi \in \mathrm{TLIM}$ (see [4, p. 50]). If G is compact then $\lambda(O^{-1}) = \lambda(O) < \frac{1}{2}$. Let $B = G \sim O^{-1}$. Then B is closed nowhere dense, $\Psi(B) = 1$ if $\Psi \in \mathrm{TLIM}$ and G is not compact while $\lambda(B) > \frac{1}{2}$ if G is compact. (In this last case $\{\lambda\} = \mathrm{TLIM} = \mathrm{TRIM}$.) In different terminology B is topologically left almost convergent to 1 (or to a positive real $> \frac{1}{2}$ if G is compact).

We claim that $\phi(B) = 0$ for some $\phi \in \mathrm{LIM}$. If not, then

$$m(1_B) = \inf\{\phi(1_B);\ \phi \in \mathrm{LIM}\} = \sup_{\mathscr{A}} \operatorname{ess\,inf}_x \frac{1}{n} \sum_1^n 1_B(a_i x) = d > 0.$$

But then, there are $b_1, \cdots, b_k$ in G such that $\operatorname{ess\,inf}_x k^{-1} \sum_1^k 1_B(b_j x) \geqq d/2$ i.e. locally a.e. in x one has $k^{-1} \sum_1^k 1_{b_j^{-1}B}(x) \geqq d/2 > 0$. But this contradicts the fact that $A = G \sim \bigcup_1^k b_j^{-1} B$ is open dense, hence of nonzero Haar measure and for $x \in A$, $k^{-1} \sum_1^k 1_B(b_j x) = 0$. Using Remark 3 above one could easily show that in fact $\phi(B) = 0$ for some two sided invariant mean ϕ on $L^\infty(G)$.

REMARKS. Let G be a locally compact amenable group with $G_0 \subset G$ an open subgroup. Let λ (λ_0) be the Haar measures on G (G_0). As known and easily shown the λ_0 measurable sets comprise exactly the λ measurable sets of G which are included in G_0. We can and shall choose λ_0 to be the restriction of λ to G_0. (We use the terminology of [7].)

For $f \in L^\infty(G)$ define $(\pi f)(x) = f(x)$ for $x \in G_0$. Then π can be considered as a map onto $L^\infty(G_0)$. If $\nu \in L^\infty(G_0)^*$ is left invariant and $f \in L^\infty(G)$, let $(S_\nu f)(z) = \nu(\pi l_z f)$ for all $z \in G$. Let $\{z_\alpha G_0\}_{\alpha \in I}$ be a fixed decomposition of G into left cosets w.r.t. G_0. Then, the bounded function, $S_\nu f$ is constant on each $z_\alpha G_0$ (as known) since if $z = z_\alpha a$, $a \in G_0$ then $S_\nu f(z_\alpha a) = \nu(\pi l_{z_\alpha a} f) = \nu(l_a(\pi l_{z_\alpha} f)) = S_\nu f(z_\alpha)$, since $a \in G_0$. This implies that $S_\nu f \in UCB_l(G)$ (i.e. is left uniformly continuous as in [7] for all $f \in L^\infty(G)$ and left invariant $\nu \in L^\infty(G_0)^*$. This is the case since for all $z \in G$, $x \in G_0$, $S_\nu f(zx) - S_\nu f(z) = 0$ and G_0 is open.

Choose and fix now some LIM, μ_0 on $C(G)$ and define for any left invariant $\nu \in L^\infty(G_0)^*$, $T\nu \in L^\infty(G)^*$, by $T\nu(f) = \mu_0(S_\nu f)$.

As known and readily checked T maps the set of left invariant elements [LIM] of $L^\infty(G_0)^*$ into the set of left invariant elements [LIM] of $L^\infty(G)^*$. The above is a refinement of a construction due to M. M. Day [**1**, p. 533]. In the above context we have the

PROPOSITION 4. *Let G be a locally compact amenable group and $G_0 \subset G$ an open normal subgroup.*

If $T\nu \in TLIM(G)$ for some $\nu \in LIM(G_0)$ then $\nu \in TLIM(G_0)$.

PROOF. If $f \in L^\infty(G_0)$ denote by f_1 its $\{z_\alpha\}$ periodic extension i.e. $f_1(z_\alpha x) = f(x)$ for all $x \in G_0$ and all α. (Note that $\{z_\alpha\}$ are fixed.) It is clear that f_1 is measurable (since it needs to be so only on compacta [7, p. 131], and G_0 is open).

If $z \in z_\alpha G_0$ then

$$(*) \qquad S_\nu(f_1)(z) = S_\nu(f_1)(z_\alpha) = \nu(\pi l_{z_\alpha} f_1) = \nu(f)$$

since if $x \in G_0$ then $(\pi l_{z_\alpha} f_1)(x) = f_1(z_\alpha x) = f(x)$. Thus $(T\nu)f_1 = \mu_0(S_\nu f_1) = \mu_0(\nu(f) \cdot 1_G) = \nu f$.

Fix now $\phi_0 \in P(G)$ with support included in G_0. Then for $f \in L^\infty(G_0)$ and $x \in G_0$ one has:

$$\begin{aligned} l_{z_\alpha}(\phi_0 * f_1)(x) &= \int f_1(y^{-1} z_\alpha x)\phi_0(y)\, dy \\ &= \int f_1((z_\alpha y z_\alpha^{-1})^{-1} z_\alpha x)\phi_0(z_\alpha y z_\alpha^{-1})\, \Delta(z_\alpha^{-1})\, dy \\ &= \int_{G_0} f_1(z_\alpha y^{-1} x)\phi_0(z_\alpha y z_\alpha^{-1})\, \Delta(z_\alpha^{-1})\, dy \\ &= \int_{G_0} f(y^{-1} x)\phi_0(z_\alpha y z_\alpha^{-1})\, \Delta(z_\alpha^{-1})\, dy = (\Psi_\alpha \circledast f)(x) \end{aligned}$$

where $\Psi_\alpha(y) = \phi_0(z_\alpha y z_\alpha^{-1})\, \Delta(z_\alpha^{-1})$ for $y \in G_0$, thus $\Psi_\alpha \in P(G_0)$ and where $\circledast$ stands for convolution in $L_1(G_0)$. Note, that since G_0 is normal $\phi_0(z_\alpha y z_\alpha^{-1})$ has support included in G_0.

It follows that if $z \in z_\alpha G_0$ then

$$\begin{aligned} S_\nu(\phi_0 * f_1)(z) &= S_\nu(\phi_0 * f_1)(z_\alpha) \\ &= \nu \pi l_{z_\alpha}(\phi_0 * f_1) = \nu(\Psi_\alpha \circledast f) = \nu(\phi_0 \circledast f). \end{aligned}$$

Note that we have used in the last equality only the fact that $\nu \in \mathrm{LIM}(G_0)$. From it alone, it follows (see Greenleaf [**6**, proof of Lemma 222, p. 27]) that $\nu(\phi \circledast f) = \nu(\Psi \circledast f)$ for all $\phi, \Psi \in P(G_0)$.

Hence $T\nu(\phi_0 * f_1) = \mu_0(S_\nu(\phi_0 * f_1)) = \nu(\phi_0 \circledast f)$.

But by assumption $T\nu \in \mathrm{TLIM}$. Thus $T\nu(\phi_0 * f_1)=(T\nu)f_1=\nu f$ and hence, for all $f \in L^\infty(G_0)$, $\nu(\phi_0 \circledast f)=\nu(f)$. The above remark implies that $\nu \in \mathrm{TLIM}(G_0)$ and finishes this proof.

THEOREM 2. *Let G be a locally compact group which is amenable as a discrete group. Assume that G contains a σ-compact open normal subgroup. If $LIM(G)=TLIM(G)$ then G is discrete.*

REMARK. 1. If G has equivalent left and right uniform structures then G contains a neighborhood U of the identity with compact closure such that $xUx^{-1}=U$ for all $x \in G$. Thus $G_0=\bigcup_{-\infty}^{\infty} U^n$ is normal σ-compact and open. In particular the theorem certainly holds true for all locally compact abelian groups G. It also holds true for all σ-compact G which are amenable as discrete groups (take $G=G_0$).

2. We could have assumed in this theorem that G is a locally compact amenable group and the open normal σ-compact G_0 is amenable as discrete. This however readily implies that G is amenable as discrete and we would not gain anything. (The discrete G/G_0 and G_0 with discrete topology are amenable hence so is G with discrete topology.)

PROOF. If $\mathrm{TLIM}(G)=\mathrm{LIM}(G)$ then $\mathrm{TLIM}(G_0)=\mathrm{LIM}(G_0)$ since $T\nu \in \mathrm{TLIM}(G)=\mathrm{LIM}(G)$ for all $\nu \in \mathrm{LIM}(G_0)$. Thus $\nu \in \mathrm{TLIM}(G_0)$ by the previous proposition. We use now Theorem 1 and get that G_0 is discrete. Thus if $x \in G_0$, $\{x\}$ is open in G_0 hence in G. Hence G is discrete.

The following is an interpretation of our and some known related results from the point of view of harmonic analysis on locally compact groups.

Let H $[H_c]$ denote the linear span of $\{f-l_x f;\ f \in L^\infty(G),\ x \in G\}$ $[\{f-\phi * f;\ f \in L^\infty(G),\ \phi \in P(G)\}]$. If $A \subset L^\infty(G)$ denote by $\bar{A}$ $[\bar{A}^*]$ its norm $[w^*]$ closure.

We need the following known remark whose proof uses a trick due to I. Namioka **[9]**.

REMARK. Let Ψ, Ψ_1, $\Psi_2 \in L^\infty(G)^*$, $\phi \in P(G)$ and define $(L_\phi \Psi)f=\Psi(\phi * f)$ for $f \in L^\infty(G)$. Let $\Psi_1 \vee \Psi_2=\max(\Psi_1, \Psi_2)$ in the lattice $L^\infty(G)^*$ and $\Psi^+=\Psi \vee O, \Psi^-=(-\Psi)\vee O$. If $\Psi \in L^\infty(G)^*$ satisfies $L_\phi \Psi=\Psi$ for all $\phi \in P(G)$, then so do Ψ^+ and Ψ^-: If $\phi \in P(G)$, $L_\phi(\Psi \vee O) \geqq (L_\phi \Psi \vee L_\phi O)=\Psi \vee O=\Psi^+$. So $L_\phi \Psi^+-\Psi^+ \geqq 0$. But $(L_\phi \Psi^+-\Psi^+)(1)=0$. Thus $L_\phi \Psi^+=\Psi^+$. (Same true, if L_ϕ is replaced by l_a^* for all $a \in G$.)

PROPOSITION 5. (a) *Let G be compact and infinite. Then*

$$\bar{H} \subset \bar{H}_c = \bar{H}_c^* = \bar{H}^* = \{f \in L^\infty(G);\ \lambda f = 0\}.$$

If G is abelian (or even amenable as a discrete group) then $\bar{H} \neq \bar{H}_c$.

(b) *Let G be a noncompact locally compact group. Then* $\bar{H} \subset \bar{H}_c \subset \bar{H}^* = \bar{H}_c^* = L^\infty(G)$. *Furthermore*

(i) $\bar{H}_c = L^\infty(G)$ *iff* $\bar{H} = L^\infty(G)$ *iff G is not amenable* (*i.e.* $LIM = \varnothing$).

(ii) *If G is σ-compact amenable then* $L^\infty(G)/\bar{H}_c$ *is a nonseparable Banach space.*

(iii) *If G is a σ-compact and amenable as discrete or amenable and containing such an open normal subgroup* (*in particular if G is locally compact abelian*), *then* $\bar{H} = \bar{H}_c$ *iff G is discrete.*

PROOF. (a) $\bar{H} \subset \bar{H}_c$ is due to the fact that TLIM $\subset$ LIM [**6**, p. 25], the remark above and the Hahn-Banach theorem (this part with G not necessarily compact). Thus $\bar{H}^* \subset \bar{H}_c^*$. If the inclusion were proper then there would exist some $\phi \in L_1(G)$ such that $\phi(H) = 0$ but $\phi(g) \neq 0$ for some $g \in H_c$. But then ϕ is left invariant and in $L_1(G)$ hence $\phi = c\lambda$ for some scalar $c \neq 0$. Hence $\phi(H_c) = \lambda(H_c) = 0$ which cannot be. So $\bar{H} \subset \bar{H}_c \subset \bar{H}_c^* = \bar{H}_c^* \subset \{f \in L^\infty(G);\ \lambda f = 0\}$.

That $\bar{H}_c = \{f \in L^\infty(G);\ \lambda f = 0\}$ is a consequence of Theorem 7.3, p. 360 of J. C. S. Wong [**10**] or can directly be proven. The rest of (a) is implied by the main theorem of this paper.

(b) If $\bar{H}^* \neq L^\infty(G)$ there would exist $0 \neq \phi \in L_1(G)$ such that $\phi(H) = 0$. But then ϕ is left invariant hence so are ϕ^+, ϕ^- and $\phi^+ \neq 0$ or $\phi^- \neq 0$. Assuming that $\phi^+ \neq 0$, $\mu(A) = \int_A \phi^+ \, d\lambda$ is a measure on the Borel sets of G satisfying all the conditions in Hewitt-Ross [**7**, p. 194]. Hence $\mu = c\lambda$ for some $c > 0$ (since $\mu \neq 0$).

Since $\mu(G) < \infty$, $\lambda(G) < \infty$ so G is compact. That (b)(i) holds is known and readily shown. (b)(ii) is shown as follows: If $L^\infty(G)/\bar{H}_c$ would be separable there would exist a sequence $\{f_n\} \subset L^\infty(G)$ such that (if B is the linear span of $\{f_n\}$) $\bar{H}_c + B$ is norm dense in $L^\infty(G)$ (see [**4**, p. 63]). But $\bar{H}_c = \{f \in L^\infty(G);\ \Psi(f) = 0$ for all $\Psi \in$ TLIM$\}$ Wong [**10**, p. 360]. Fix now some $\Psi_0 \in$ TLIM and let $\Psi_0 f_n = \alpha_n$. Then $\{\Psi_0\} = \{\Psi \in$ TLIM; $\Psi f_n = \alpha_n n \geq 1\}$ since any Ψ which belongs to the right side will coincide with Ψ_0 on $\bar{H}_c + B$ hence on $L^\infty(G)$. We apply now [**4**, Theorem 5, p. 53] with $K = P(G)$ hence $A = \{\Psi \in$ TLIM; $\Psi(f_n - \alpha_n) = 0\} = \{\Psi_0\}$ is norm separable. Thus G is compact. (b) (iii) is just our main theorem and the fact that $\bar{H} = \bar{H}_c$ iff LIM $=$ TLIM (by our remark above and the Hahn-Banach theorem).

MAIN CONJECTURE. Let G be any amenable locally compact group. If G is noncompact then $L^\infty(G)/\bar{H}_c$ is nonseparable. If G is nondiscrete then $\bar{H}_c/\bar{H}$ is nonseparable.

Addition. In the meantime W. Rudin sent us a preprint of a paper of his, in which he proves Theorem 2 without the assumption that (*) "G contains an open σ-compact normal subgroup", but with the assumption that G is amenable as discrete. His proof is different from ours and uses harmonic analysis type arguments. After reading his manuscript we found the following easy argument which removes the restriction (*).

PROPOSITION. *Let G_0 be an open noncompact subgroup of G, and*

$$G = \bigcup_{\alpha\in I} x_\alpha G_0, \qquad x_\alpha G_0 \cap x_\beta G_0 = \varnothing \quad \textit{if } \alpha \neq \beta.$$

If $A_0 \subset G_0$ is such that $\lambda(A_0) < \infty$ (λ—the Haar measure on G) then for all $\phi \in \mathrm{TRIM}$, $\phi(\bigcup_{\alpha\in I} x_\alpha A_0) = 0$.

PROOF. Let $B_n \subset G_0$ be compact with $\lambda(B_n) = a_n \uparrow \infty$ and let $f_n = a_n^{-1} 1_{B_n}$, $A = \bigcup_\alpha x_\alpha A_0$. Then

$$\begin{aligned} 1_A * \tilde{f}_n(x) &= a_n^{-1} \int 1_A(y) 1_{\tilde{B}_n}(y^{-1}x)\, dy \\ &= a_n^{-1}\lambda(xB_n \cap A) \leqq a_n^{-1}\lambda(xG_0 \cap A) \\ &= a_n^{-1}\lambda(x_\alpha G_0 \cap A) = a_n^{-1}\lambda(x_\alpha A_0) = a_n^{-1}\lambda(A_0), \end{aligned}$$

for some (hence all) $\alpha \in I$.

If $\phi \in \mathrm{TRIM}$ then $\phi(A) = \phi(1_A * \tilde{f}_n) \leqq a_n^{-1}\lambda(A_0) \to 0$.

REMARK. If $\Psi \in \mathrm{TLIM}$, then $\psi(\bigcup_\alpha A_0^{-1} x_\alpha^{-1}) = 0$. (See [**4**, pp. 49–50].) To remove restriction (*) on G, let G_0 be any σ-compact, noncompact, open subgroup of G, if G is noncompact, and $G = G_0$, if G is compact. Let $A_0 \subset G_0$ be open dense with $\lambda(A_0) \leqq \frac{1}{2}$ and $A = \bigcup_\alpha A_0^{-1} x_\alpha^{-1}$ (x_α as above), $A = A_0$ if G is compact. Let $B = G \sim A$. Then $\psi(B) = 1$ for all $\psi \in \mathrm{TLIM}$, if G is not compact, $\lambda(B) \geqq \frac{1}{2}$ if G is compact. B is closed nowhere dense. Continue now as in the proof of Theorem 1.

REFERENCES

1. M. M. Day, *Amenable semigroups*, Illinois J. Math **1** (1957), 509–544. MR **19**, 1067.

2. E. Følner, *On groups with full Banach mean values*, Math. Scand. **3** (1955), 243–254. MR **18**, 51.

3. ———, *Note on groups with and without full Banach mean value*, Math. Scand. **5** (1957), 5–11. MR **20** #1237.

4. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. No. 123 (1972).

5. ———, *On finite equivalent invariant measures for semigroups of transformations*, Duke Math. J. **38** (1971), 395–408. MR **44** #404.

6. F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Math. Studies, no. 16, Van Nostrand Reinhold, New York, 1969. MR **40** #4776.

7. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. I: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York, Springer-Verlag, Berlin, 1963. MR **28** #158.

8. I. Namioka, *Folner's conditions for amenable semigroups*, Math Scand. **15** (1964), 18–28. MR **31** #5062.

9. ———, *On certain actions of semigroups on L-spaces*, Studia Math. **29** (1967), 63–77. MR **36** #6910.

10. J. C. S. Wong, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc. **144** (1969), 351–363. MR **40** #2781.

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