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ON AKCOGLU AND SUCHESTON'S OPERATOR CONVERGENCE THEOREM IN LEBESGUE SPACE

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ABSTRACT. Let T be a bounded linear operator on an L_1-space and τ its linear modulus. It is proved that if the adjoint of τ has a strictly positive subinvariant function then the following two conditions are equivalent: (i) T^n converges weakly; (ii) $(1/n)\sum_{i=1}^n T^{k_i}$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.

1. **Introduction.** Let $(X, \mathscr{M}, m)$ be a σ-finite measure space and $L_p(X)=L_p(X, \mathscr{M}, m)$, $1\leqq p\leqq\infty$, the usual (complex) Banach spaces. If $A \in \mathscr{M}$ then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X-A$. Let T be a bounded linear operator on $L_1(X)$ and τ its linear modulus [2]. Thus τ is a positive linear operator on $L_1(X)$ such that

$$\|\tau\|_1 = \|T\|_1 \quad \text{and} \quad \tau g = \sup\{|Tf|; f\in L_1(X) \text{ and } |f|\leqq g\}$$

for any $0\leqq g \in L_1(X)$. The adjoint of T is denoted by T^*. Clearly T is a contraction if and only if $\tau^*1\leqq 1$. In [1] Akcoglu and Sucheston proved that if T is a contraction then the following two conditions are equivalent: (i) T^n converges weakly; (ii) $(1/n)\sum_{i=1}^n T^{k_i}$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers. In this note we shall prove that if τ^* has a strictly positive subinvariant function in $L_\infty(X)$ then the equivalence of (i) and (ii) still holds. Applying this result, we obtain that if T is a *positive* linear operator on $L_1(X)$ such that $\sup_n \|(1/n)\sum_{k=0}^{n-1} T^k\|_1<\infty$ and also such that $T^n f$ converges weakly for any $f\in L_1(X)$ with $\int f\,dm=0$ and if T^* has a strictly positive subinvariant function in $L_\infty(X)$, then for any $f\in L_1(X)$ with $\int f\,dm=0$ and any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers, $(1/n)\sum_{i=1}^n T^{k_i} f$ converges strongly. This is a generalization of another result of Akcoglu and Sucheston [1].

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2. **Results.** Throughout this section we shall assume that there exists a strictly positive function $s \in L_\infty(X)$ with $\tau^* s \leqq s$. In the proofs we shall also assume that $(X, \mathcal{M}, m)$ is a finite measure space, since the L_1 of a σ-finite measure space is isometric to the L_1 of a finite measure space (cf. [1]).

THEOREM 1. *The following two conditions are equivalent*:

(i) *If $f \in L_1(X)$ then $T^n f$ converges weakly*;

(ii) *If $f \in L_1(X)$ then $(1/n)\sum_{i=1}^n T^{k_i} f$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.*

PROOF. We first prove that (i) implies (ii). For $sf \in L_1(X)$, where $f \in L_1(X)$, define $V(sf) = sTf$. Since $\{sf; f \in L_1(X)\}$ is a dense subspace of $L_1(X)$ in the norm topology and $\|V(sf)\|_1 \leqq \|sf\|_1$ (cf. [3]), V may be considered to be a linear contraction on $L_1(X)$. Since $V^n(sf) = sT^n f$ for any $n \geqq 0$ and $T^n f$ converges weakly, it follows that $V^n(sf)$ converges weakly. Thus, since V is a contraction, it is easily seen that for any $A \in \mathcal{M}$ the limit $\mu(A) = \lim_n \int_A V^n f\, dm$ exists. Since the measure m is finite, the Vitali-Hahn-Saks theorem implies that μ is a countably additive measure on $\mathcal{M}$ absolutely continuous with respect to m. Therefore there exists a function $g \in L_1(X)$ such that $\mu(A) = \int_A g\, dm$ for any $A \in \mathcal{M}$. It follows that $V^n f$ converges weakly to g. Thus, by Theorem 2.1 of [1], for any $f \in L_1(X)$ and any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers,

$$\frac{1}{n}\sum_{i=1}^n V^{k_i}(sf) = \frac{1}{n} s\left(\sum_{i=1}^n T^{k_i} f\right)$$

converges strongly. Let $\lim_n \|(1/n)s(\sum_{i=1}^n T^{k_i} f) - f_0\|_1 = 0$ for some $f_0 \in L_1(X)$ and let $\varepsilon > 0$ be arbitrarily fixed. Since $T^n f$ converges weakly, there exists a positive number δ such that $A \in \mathcal{M}$ and $m(A) < \delta$ imply $\int_A |T^n f|\, dm < \varepsilon$ for any $n \geqq 0$. Choose $\eta > 0$ such that $m(\{x; s(x) < \eta\}) < \delta$ and $\int_{\{x; s(x) < \eta\}} |f_0|\, dm < \varepsilon$, and put $A = \{x; s(x) < \eta\}$. Then

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{i=1}^n T^{k_i} f - \frac{1}{m}\sum_{j=1}^m T^{k_j} f\right\|_1 &\leqq \left\|\frac{1}{n}\sum_{i=1}^n 1_A T^{k_i} f\right\|_1 + \left\|\frac{1}{m}\sum_{j=1}^m 1_A T^{k_j} f\right\|_1 \\
&\quad + \left\|\frac{1}{n}\sum_{i=1}^n 1_{X-A} T^{k_i} f - \frac{1}{m}\sum_{j=1}^m 1_{X-A} T^{k_j} f\right\|_1 \\
&< 2\varepsilon + \left\|\frac{1}{n}\sum_{i=1}^n 1_{X-A} T^{k_i} f - 1_{X-A}\frac{1}{s} f_0\right\|_1 \\
&\quad + \left\|\frac{1}{m}\sum_{j=1}^m 1_{X-A} T^{k_j} f - 1_{X-A}\frac{1}{s} f_0\right\|_1
\end{aligned}$$

and

$$\left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i} f - 1_{X-A} \frac{1}{s} f_0 \right\|_1$$
$$\leqq \frac{1}{\eta} \left\| \frac{1}{n} s\left(\sum_{i=1}^{n} 1_{X-A} T^{k_i} f\right) - 1_{X-A} f_0 \right\|_1 \to 0$$

as $n \to \infty$, from which we observe that $(1/n) \sum_{i=1}^{n} T^{k_i} f$ is a Cauchy sequence in $L_1(X)$, and hence $(1/n) \sum_{i=1}^{n} T^{k_i} f$ converges strongly.

Conversely if (ii) holds, then it follows easily that $\sup_n \|T^n\|_1 < \infty$ and that for any $f \in L_1(X)$ and any $A \in \mathcal{M}$, $\lim_n \int_A T^n f\, dm$ exists, and hence $T^n f$ converges weakly. This completes the proof of Theorem 1.

THEOREM 2. *Let T be a positive linear operator on $L_1(X)$ with*

$$\sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\|_1 < \infty$$

and suppose $T^* s \leqq s$ for some $0 < s \in L_\infty(X)$. Then the following two conditions are equivalent:

(i) *If $f \in L_1(X)$ and $\int f\, dm = 0$, then $T^n f$ converges weakly*;

(ii) *If $f \in L_1(X)$ and $\int f\, dm = 0$, then for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers, $(1/n) \sum_{i=1}^{n} T^{k_i} f$ converges strongly.*

PROOF. Suppose (i) holds. It is known [3] that if T has no nontrivial nonnegative invariant function in $L_1(X)$, then the operator V introduced above also has no nontrivial nonnegative function in $L_1(X)$. Thus it follows from [1] that, if $T^n f$ converges weakly then

$$\lim_n \|V^n(sf)\|_1 = \lim_n \|sT^n f\|_1 = 0.$$

Let $\varepsilon > 0$ be arbitrarily fixed, and let δ be a positive number such that $A \in \mathcal{M}$ and $m(A) < \delta$ imply $\int_A |T^n f|\, dm < \varepsilon$ for any $n \geqq 0$. Choose $\eta > 0$ such that $m(\{x; s(x) < \eta\}) < \delta$, and put $A = \{x; s(x) < \eta\}$. Then

$$\|T^n f\|_1 \leqq \|1_A T^n f\|_1 + \eta^{-1} \|1_{X-A} s T^n f\|_1$$
$$< \varepsilon + \eta^{-1} \|sT^n f\|_1$$

and $\|sT^n f\| \to 0$ as $n \to \infty$, thus $\lim_n \|T^n f\|_1 = 0$.

If there exists $0 \leqq h \in L_1(X)$ with $\|h\|_1 > 0$ and $Th = h$, then it follows from [1] that for any $f \in L_1$, $T^n f$ converges weakly. Thus the strong convergence of $(1/n) \sum_{i=1}^{n} T^{k_i} f$ for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers follows from Theorem 1.

Clearly (ii) implies (i), and the proof is complete.

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