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A GENERALIZATION OF TIETZE'S THEOREM ON CONVEX SETS IN R^3

NICK M. STAVRAKAS

ABSTRACT. Let $S\subset R^3$ and let $C(S)$ denote the points of local convexity of S. One interesting result which is proven is

THEOREM. *Let $S\subset R^3$ be such that $S\subset \mathrm{cl}(C(S))$, S not planar and $C(S)$ is connected. Then $S\subset \mathrm{cl}(\mathrm{int}\, S)$.*

1. **Introduction.** F. A. Valentine in [8] proves that if S is a closed connected subset of R^d whose points of local nonconvexity are decomposable into n convex sets, then S is $2n+1$ polygonally connected. Guay and Kay in [2] show that if S is a closed connected subset of a topological vector space such that S has exactly n points of local nonconvexity and such that the points of local convexity of S are connected, then S is expressible as a union of $n+1$ or fewer closed convex sets. The purpose of this paper is to give a result which is in the vein of both the latter mentioned results and which generalizes Tietze's theorem on convex sets in R^3. For related results see [1], [2], [3], [4], [5], [6] and [8].

2. **Notations and main results.** If $S\subset R^d$, the symbols $C(S)$ and $L(S)$ denote the points of local convexity of S and points of local nonconvexity of S, respectively. The symbols int S and cl S denote the interior of S and the closure of S, respectively.

THEOREM 1. *Let $S\subset R^3$ be such that*
(1) $S\subset \mathrm{cl}(C(S))$,
(2) *S not planar*,
(3) *$C(S)$ is connected.*
Then $S\subset \mathrm{cl}(\mathrm{int}\, S)$.

PROOF. We first prove $C(S)\subset \mathrm{cl}(\mathrm{int}\, S)$. Suppose not. Then there exists $x\in C(S)$ and an open set M_x about x such that $M_x\cap S$ is convex and $\dim(M_x\cap S)=k<3$. Let L be the subspace generated by $M_x\cap S$. Let $\mathscr{M}=\{M \mid M$ is open in $L\cap S$, $M_x\cap S\subset M$ and if $y\in M$, there exists an open set N_y about y such that $N_y\cap S$ is convex and $\dim(N_y\cap S)=k\}$. Note $\mathscr{M}\neq\varnothing$ since $M_x\cap S\in\mathscr{M}$. Partially order $\mathscr{M}$ by set inclusion. Using a standard

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Zorn's lemma argument, it may be shown $\mathcal{M}$ has a maximal element A. Since S is not planar, there exists $z \in S$, with $z \notin L$. Select a point q as follows: If $z \in C(S)$, set $z=q$. If $z \in L(S)$, since $S \subset \mathrm{cl}(C(S))$, there exists a point $r \in C(S)$, with $r \notin L$. Then set $q=r$. Since $C(S)$ is connected and locally convex, $C(S)$ is polygonally connected. Let l be a simple polygonal arc from x to q in $C(S)$. Regarding x as the starting point of l, let m be the last point of l in cl A. Since $l \subset C(S)$, there exists an open set N_m such that $N_m \cap S$ is convex. It is clear that $\dim(N_m \cap S) \geqq k$. We consider two cases.

Case 1. $\dim(N_m \cap S)=k$. Then $N_m \cap S \subset L$ and since $N_m \cap S$ contains points of l not in A, we have $N_m \cap A \subsetneqq N_m \cap S$. Then $A \cup (N_m \cap S) \in \mathcal{M}$, contradicting the maximality of A.

Case 2. $\dim(N_m \cap S)>k$. Now since $N_m \cap A \neq \varnothing$, we may choose $p \in N_m \cap A$. Then for any open set N_p such that $N_p \cap S$ is convex, $\dim(N_p \cap S) \geqq \dim(N_m \cap S)>k$, contradicting that $A \in \mathcal{M}$.

Thus $C(S) \subset \mathrm{cl}(\mathrm{int}\, S)$ and the latter with hypothesis (1) imply the Theorem.

The following theorem is the main result of this paper.

THEOREM 2. *Let $S \subset R^3$ be closed, S not planar. Suppose $L(S)$ decomposable into n closed line segments $[a_i b_i]$, $1 \leqq i \leqq n$. Suppose $C(S)$ is connected and that given $x, y \in C(S)$ that x and y may be joined by an arc $l \subset S$ such that l is contained in a hyperplane. Then S is $n+1$ polygonally connected.*

PROOF. The fact that $L(S)$ is decomposable into n closed line segments easily implies that $S \subset \mathrm{cl}(C(S))$. Let $x, y \in S$ and let $\mathcal{H}_{xy}$ denote the set of all hyperplanes containing x and y. Define a set F by $F=\{(x, y) | (x, y) \in C(S) \times C(S)$ and if $H_{xy} \in \mathcal{H}_{xy}$, $\dim(H_{xy} \cap [a_i b_i]) \leqq 0$ $\forall i$, $0 \leqq i \leqq n\}$, where in the definition of F we take $\dim \varnothing = -1$. Let $(x, y) \in F$. Then by hypothesis there exists $H_{xy} \in \mathcal{H}_{xy}$ and an arc $l \subset S$ from x to y such that $l \subset H_{xy}$. Let C be the component of $H_{xy} \cap S$ which contains x and y. Since $\dim(H_{xy} \cap [a_i b_i]) \leqq 0$, $\forall i$, C has at most n points of local nonconvexity and by a result of Valentine [8], C is $n+1$ polygonally connected. Thus x and y may be joined by an $n+1$ polygonal arc lying in S. By Theorem 1, F is dense in $S \times S$, and the theorem follows from a standard limiting argument in the Hausdorff metric.

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DEPARTMENT OF MATHEMATICS, CLEMSON UNIVERSITY, CLEMSON, SOUTH CAROLINA 29631