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ANALYTIC FUNCTIONS, IDEALS, AND DERIVATION RANGES[1]

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ABSTRACT. When A is in the Banach algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$, the derivation generated by A is the bounded operator Δ_A on $\mathscr{B}(\mathscr{H})$ defined by $\Delta_A(X)=AX-XA$. It is shown that (i) if B is an analytic function of A, then the range of Δ_B is contained in the range of Δ_A; (ii) if U is a nonunitary isometry, then the range of Δ_U contains nonzero left ideals; (iii) if U and V are isometries with orthogonally complemented ranges, then the span of the ranges of the corresponding derivations is all of $\mathscr{B}(\mathscr{H})$.

1. It follows from the elementary properties of derivations that the set of all B such that $\mathscr{R}(\Delta_B)\subset\mathscr{R}(\Delta_A)$ is a subalgebra of $\mathscr{B}(\mathscr{H})$. (See [**9**, p. 4].) Therefore if B is a polynomial in A, then $\mathscr{R}(\Delta_B)\subset\mathscr{R}(\Delta_A)$. We will generalize this to analytic functions. In the following, $\mathscr{H}$ denotes a separable complex Hilbert space.

THEOREM 1. *Let $A\in\mathscr{B}(\mathscr{H})$ and let $f(z)$ be a function analytic on an open set containing $\sigma(A)$. If $B=f(A)$, then $\mathscr{R}(\Delta_B)\subset\mathscr{R}(\Delta_A)$.*

For the proof we need the following result on analytic functions of commuting operators.

Let $\mathscr{A}$ be a commutative Banach algebra with maximal ideal space $\mathscr{M}_{\mathscr{A}}$ and let a_1 and a_2 belong to $\mathscr{A}$. The joint spectrum of a_1 and a_2 is the set $\{(\varphi(a_1), \varphi(a_2)) : \varphi\in\mathscr{M}_{\mathscr{A}}\}$ and is denoted by $\sigma(a_1, a_2)$. (See Gamelin [**3**, p. 76] for a discussion of the joint spectrum and the proof of the following lemma.)

LEMMA 1. *There exists a unique rule assigning to every ordered pair (a_1, a_2) of elements in $\mathscr{A}$ and to every complex valued function of two complex variables $f(z, w)$ analytic in a neighborhood of $\sigma(a_1, a_2)$, an element*

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$f(a_1, a_2) \in \mathscr{A}$ satisfying the following conditions:

(a) *If $f(z, w) = \sum c_i d_j z^i w^j$ is a polynomial, then $f(a_1, a_2) = \sum c_i d_j a_1^i a_2^j$.*

(b) *If $f(z, w)$ and $g(z, w)$ are analytic in a neighborhood of $\sigma(a_1, a_2)$, then*

$$(f+g)(a_1, a_2) = f(a_1, a_2) + g(a_1, a_2)$$

and

$$(fg)(a_1, a_2) = f(a_1, a_2)g(a_1, a_2).$$

(c) *If $f(z)$ is analytic in a neighborhood U of $\sigma(a_1)$ and if $f_1(z, w)$ is the extension of $f(z)$ to $U \times \mathscr{C}$ defined by $f_1(z, w) = f(z)$, then $f_1(a_1, a_2) = f(a_1)$ where $f(a_1)$ is an analytic function of the element a_1 in the sense of the Riesz-Dunford functional calculus (Dunford and Schwartz* [**2**, *p.* 566]).

PROOF OF THEOREM 1. For $A \in \mathscr{B}(\mathscr{H})$, let L_A and R_A be the operators on $\mathscr{B}(\mathscr{H})$ defined by $L_A(X) = AX$ and $R_A(X) = XA$. It is not difficult to show that $\sigma(L_A) = \sigma(R_A) = \sigma(A)$. Therefore if $f(z)$ is analytic on a neighborhood of $\sigma(A)$, then both $f(L_A)$ and $f(R_A)$ are defined by the usual Riesz-Dunford functional calculus. Furthermore, it is known [**5**, p. 33] that $f(L_A) = L_{f(A)}$ and $f(R_A) = R_{f(A)}$. Let $\mathscr{A}$ be the maximal abelian subalgebra of $\mathscr{B}(\mathscr{B}(\mathscr{H}))$ containing L_A, R_A, and the identity. Then the spectrum of L_A (and R_A) with respect to the algebra $\mathscr{A}$ is equal to the spectrum of L_A (and R_A) with respect to the algebra $\mathscr{B}(\mathscr{B}(\mathscr{H}))$ which is $\sigma(A)$. (See [**7**, p. 34].) We will apply Lemma 1 to the commutative algebra $\mathscr{A}$. If $g(z, w) = (f(z) - f(w))/(z - w)$, then it can easily be shown that $g(z, w)$ is analytic on a neighborhood of $\sigma(L_A, R_A)$. Let $h(z, w) = (z - w)g(z, w)$. Then by Lemma 1 part (b) there exists an operator $h(L_A, R_A)$ in $\mathscr{A}$ such that $h(L_A, R_A) = f(L_A) - f(R_A)$ and by parts (a) and (b) $h(L_A, R_A) = (L_A - R_A)g(L_A, R_A)$. Therefore $f(L_A) - f(R_A) = (L_A - R_A)g(L_A, R_A)$. Hence

$$\Delta_{f(A)} = L_{f(A)} - R_{f(A)} = f(L_A) - f(R_A) = \Delta_A g(L_A, R_A)$$

and therefore $\mathscr{R}(\Delta_{f(A)}) \subset \mathscr{R}(\Delta_A)$.

COROLLARY 1. *Let $A \geqq 0$ be an element of $\mathscr{B}(\mathscr{H})$ with $0 \notin \sigma(A)$. Then $\mathscr{R}(\Delta_{A^{1/2}}) = \mathscr{R}(\Delta_A)$.*

PROOF. Since the function $f(z) = z^{1/2}$ is analytic on the right half plane, $\mathscr{R}(\Delta_{A^{1/2}}) \subset \mathscr{R}(\Delta_A)$. The reverse inclusion follows from the fact that $A = (A^{1/2})^2$.

2. Stampfli [**8**] has shown that the range of a derivation does not contain any nonzero two-sided ideals. We will see that the range of a derivation generated by a nonunitary isometry does contain nonzero left ideals.

REMARK. If U is a pure isometry and $\mathscr{D}=\mathscr{R}(U)^{\perp}$, then it can be shown that all operators of the form

$$A=\begin{bmatrix} A_0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ A_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ A_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \end{bmatrix}$$

on $\mathscr{H}=\mathscr{D}\oplus U(\mathscr{D})\oplus U^2(\mathscr{D})\oplus\cdots$ are in $\mathscr{R}(\Delta_U)$. (See Pearcy [**6**] or Halmos [**4**].) It is an immediate consequence that $\mathscr{B}(\mathscr{H})(1-UU^*)\subset \mathscr{R}(\Delta_U)$. This result can be extended to all isometries.

THEOREM 2. *Let U be an isometry on $\mathscr{H}$. If $P=1-UU^*$, then $\mathscr{B}(\mathscr{H})P\subset\mathscr{R}(\Delta_U)$.*

PROOF. Let $U=V\oplus W$ on $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$ where V is a pure isometry, W is a unitary, and $\mathscr{H}_i$ is an infinite dimensional Hilbert space for $i=1, 2$. Given $X\in\mathscr{B}(\mathscr{H})$ where

$$X=\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

choose $Y_1\in\mathscr{B}(\mathscr{H}_1)$ such that $X_1(1-VV^*)=VY_1-Y_1V$ (the existence of which is guaranteed by the above remark). If we let

$$Y=\begin{bmatrix} Y_1 & 0 \\ W^*X_3(1-VV^*) & 0 \end{bmatrix}$$

then a computation shows that $X(1-UU^*)=\Delta_U(Y)$.

REMARKS. (1) A more algebraic proof can be obtained by seeing that for $Y\in\mathscr{B}(\mathscr{H})$, the operator $X=\sum_{k=0}^{\infty} U^kPYPU^{*k+1}$ is bounded and that $\Delta_U(U^*YP-X)=YP$.

(2) Let $U=V\oplus W$ on $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$ where V and W are both required only to be isometries. For $Y=(Y_i)\in\mathscr{B}(\mathscr{H})$

$$\Delta_U(Y)=\begin{bmatrix} VY_1-Y_1V & VY_2-Y_2W \\ WY_3-Y_3V & WY_4-Y_4W \end{bmatrix}$$

and for $X=(X_i)\in\mathscr{B}(\mathscr{H})$

$$X(1-UU^*)=\begin{bmatrix} X_1(1-VV^*) & X_2(1-WW^*) \\ X_3(1-VV^*) & X_4(1-WW^*) \end{bmatrix}.$$

Since U is an isometry, $\mathscr{B}(\mathscr{H})(1-UU^*)\subset\mathscr{R}(\Delta_U)$ by Theorem 2. By considering the (2, 1) positions in the above matrices, it follows that given any bounded operator $X:\mathscr{H}_1\to\mathscr{H}_2$, there exists a bounded operator $Y:\mathscr{H}_1\to\mathscr{H}_2$ such that $X(1-VV^*)=WY-YV$. In particular, if V and W are isometries on $\mathscr{H}_1$ and W_1 is a unitary from $\mathscr{H}_1$ onto $\mathscr{H}_2$, then $V\oplus W_1WW_1^*$ is an isometry on $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$. Therefore, for each $X\in\mathscr{B}(\mathscr{H}_1)$ there exists a $Y\in\mathscr{B}(\mathscr{H}_1)$ such that $W_1X(1-VV^*)=W_1WW_1^*(W_1Y)-(W_1Y)V$. Therefore $X(1-VV^*)=WY-YV$.

COROLLARY. *If V and W are isometries on $\mathscr{H}$, then $\mathscr{B}(\mathscr{H})(1-VV^*)$ is contained in the range of the intertwining operator $T(X)=WX-XV$.*

REMARKS. (1) By the use of Theorem 2 we can show that $\mathscr{R}(\Delta_U)$ contains other left ideals, in fact $\mathscr{B}(\mathscr{H})(1-U_\lambda U_\lambda^*)\subset\mathscr{R}(\Delta_U)$ for $U_\lambda=(U-\lambda)(1-\bar{\lambda}U)^{-1}$. To obtain an operator such that its derivation range contains right ideals, we need only consider the adjoint of a nonunitary isometry.

(2) The right ideal generated by $1-UU^*$ is not contained in $\mathscr{R}(\Delta_U)$. (See [**9**].)

3. It was observed by Halmos [**4**] that every operator on an infinite dimensional Hilbert space is the sum of two commutators. This result can be strengthened.

THEOREM 3. *Let U and V be isometries on an infinite dimensional Hilbert space. If $\mathscr{R}(U)\oplus\mathscr{R}(V)=\mathscr{H}$, then $\mathscr{R}(\Delta_U)+\mathscr{R}(\Delta_V)=\mathscr{B}(\mathscr{H})$.*

PROOF. Let $P_1=1-UU^*$ and $P_2=1-VV^*$. Then for $X\in\mathscr{B}(\mathscr{H})$, $X=XP_1+XP_2$. Hence $X\in\mathscr{R}(\Delta_U)+R(\Delta_V)$ by Theorem 2.

REMARK. Although Stampfli [**8**] has shown that $\mathscr{R}(\Delta_A)$ cannot be dense in $\mathscr{B}(\mathscr{H})$, Theorem 3 shows that $\mathscr{R}(\Delta_U)+\mathscr{R}(\Delta_V)$ is dense if U and V are the isometries $U:\mathscr{H}\to\mathscr{M}$ and $V:\mathscr{H}\to\mathscr{M}^\perp$ associated with any infinite dimensional subspace $\mathscr{M}$ of infinite deficiency.

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