MORE ON TIGHT ISOMETRIC IMMERSIONS OF CODIMENSION TWO

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ABSTRACT. We continue our investigation on tight isometric immersion of a nonnegatively curved compact manifold M^n into R^{n+2} . Under some minor restrictions, we prove that the immersion is a product embedding of convex hypersurfaces. For surfaces in R^4 , the restrictions are unnecessary.

1. Introduction. In [1], we investigated tight isometric immersions of a nonnegatively curved compact manifold M^n into R^{n+2} . We proved that the Morse number $\mu(M)$ of M cannot exceed four, and we tried to characterize all such immersions with $\mu(M)=4$. The result of Theorem C is that under certain conditions M is topologically a product of spheres. The purpose here is to prove that the product is in fact a Riemannian product.

The main reason that we could not conclude that M is a Riemannian product in [1] is that we did not check the angle between our complementary foliations on M . In this paper we will first express some results in [1] in terms of forms and then prove the result by using the structure equations. B. Hempstead [4] has proved a similar result for $n=2$ under a stronger hypothesis. For $n=2$, we will prove that only tightness and flatness are needed to prove the main result.

2. Invariant method and Cartan method in differential geometry. Let $f: M^n \rightarrow R^{n+k}$ be an isometric immersion of an n -dimensional Riemannian manifold into $(n+k)$ -dimensional euclidean space. The connections ∇ of M^n and $\bar{\nabla}$ of R^{n+k} are related as follows: For any two vector fields x, y tangent to M , $\nabla_x y$ equals the component of $\bar{\nabla}_x y$ normal to the tangent space TM of M . The difference $T_x y = \bar{\nabla}_x y - \nabla_x y$ is called the shape operator of M^n in R^{n+k} , and the second fundamental form with respect to a normal vector z on M is a linear transformation defined by $\langle S_z(x), y \rangle = -\langle T_x y, z \rangle$, where x, y are tangents to M and $\langle \cdot, \cdot \rangle$ is the scalar product in R^{n+k} .

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On the other hand, consider the bundle of orthonormal frames $m e_1, \dots, e_{n+k}$, where $m \in M$, $e_1, \dots, e_n$ are tangent vectors to M at m , and $e_{n+1}, \dots, e_{n+k}$ are normal to M at m . Define the usual differential forms $\omega_A = dm \cdot e_A$ and $\omega_{AB} = de_A \cdot e_B$. We agree in this section on the following ranges of indices:

$$1 \leq a, b, c \leq n, \quad n+1 \leq r, s, t \leq n+k, \quad 1 \leq A, B, C \leq n+k.$$

Restricting all forms to M , we have $\omega_r = 0$, and the first structure equations $d\omega_r = \sum_a \omega_a \wedge \omega_{ar} = 0$. Since the ω_a are linearly independent, we have

$$(1) \quad \omega_{ra} = \sum_b \xi_{r ab} \omega_b$$

where the $\xi_{r ab}$ are symmetric in a, b :

$$(2) \quad \xi_{r ab} = \xi_{r ba}.$$

The matrices $(\xi_{r ab})$ are the matrices of the linear transformation S_{e_r} with respect to the base $e_1, \dots, e_n$. Moreover, we have

$$(3) \quad T_{e_a} x = T_x e_a = \sum_r \omega_{ar}(x) e_r$$

for any x tangent to M . Finally, we wish to recall the second structure equations:

$$(4) \quad d\omega_{AB} = \omega_{AB} \wedge \omega_{BC}.$$

3. Tight isometric immersion of M^n into R^{n+2} . The total curvature of an immersed manifold $f: M^n \rightarrow R^{n+2}$ is defined to be ([1], [3], [5])

$$\tau(M) = \frac{1}{C_{n+1}} \int_B |\det S_z| |w|,$$

where C_{n+1} denotes the volume of the unit $(n+1)$ -sphere, $|w|$ denotes the volume element of the unit normal bundle B on M^n in R^{n+2} , and the integration is over all $z \in B$. It is known that $\tau(M)$ is bounded below by the Morse number $\mu(M)$ of M , and f is called tight if $\mu(M) = \tau(M)$. Let M^n be a compact manifold of nonnegative curvature and $f: M^n \rightarrow R^{n+2}$ be a tight isometric immersion. In [1] we proved that $\mu(M) \leq 4$; for the case $\mu(M) = 4$ we assumed that $T_x x \neq 0$ for $x \neq 0$. Under these assumptions, we showed (Lemma 3.5 of [1]) that M^n is foliated by leaves U_1 of codimension q and simultaneously by leaves U_2 of codimension p where $p+q=n$. Moreover, [1, Lemma 3.1, Lemma 3.5], we have proved that there is a

normal frame field f_1, f_2 such that

(5) x tangent to leaf of U_1 if and only if $T_x y$ is parallel to f_2 for all y .

(6) x tangent to leaf of U_2 if and only if $T_x y$ is parallel to f_1 for all y .

(7) $\bar{\nabla}_x f_1 = 0$ for x tangent to leaf of U_1 .

(8) $\bar{\nabla}_x f_2 = 0$ for x tangent to leaf of U_2 .

For each $m \in M^n$ pick a local frame $e_1, \dots, e_{p+q}$ such that $e_1, \dots, e_p$ are tangent to leaf L_1 of U_1 . Let $e'_{p+1}, \dots, e'_{p+q}$ be a frame tangent to leaf L_2 of U_2 ; we infer that

$$e'_\alpha = \sum_{A=1}^n a_{\alpha A} e_A, \quad p+1 \leq \alpha \leq p+q.$$

Throughout this section, we agree on the following index convention:

$$1 \leq i, j, k \leq p, \quad p+1 \leq \alpha, \beta, \gamma \leq p+q, \quad 1 \leq A, B, C \leq p+q,$$

and the normals e_{n+1}, e_{n+2} of the last section are replaced by respectively f_1, f_2 .

Let ω_A be the dual forms and $\omega_{AB}, \omega_{A \ n+1}, \omega_{A \ n+2}$ be the connection forms. By (1), let $\omega_{n+i \ A} = \sum_B \xi_{n+i \ AB} \omega_B$, $\xi_{n+i \ AB} = \xi_{n+i \ BA}$, $i=1, 2$. By (5), $T_{e_i} y$ is parallel to f_2 for all y . By (3)

$$T_{e_i} y = T_y e_i = \omega_{i \ n+1}(y) f_1 + \omega_{i \ n+2}(y) f_2.$$

Hence

$$(9) \quad \omega_{i \ n+1} = 0.$$

By (7), $\bar{\nabla}_{e_i} f_1 = 0$. In terms of ω 's,

$$(10) \quad \omega_{n+1, n+2}(e_i) = 0.$$

Similarly, (8) will give $\omega_{n+1, n+2}(e'_\alpha) = 0$ or $\sum_A a_{\alpha A} \omega_{n+1, n+2}(e_A) = 0$. By (10), it reduces to $\sum_\beta a_{\alpha \beta} \omega_{n+1, n+2}(e_\beta) = 0$. Since $(a_{\alpha \beta})$ is nonsingular, we must have $\omega_{n+1, n+2}(e_\beta) = 0$ or

$$(11) \quad \omega_{n+1, n+2} = 0.$$

Using the structure equations (4) and (9), we find $0 = d\omega_{n+1, n+2} = \sum_A \omega_{n+1 \ A} \omega_{A \ n+2} = \sum_\alpha \omega_{n+1 \ \alpha} \omega_{\alpha \ n+2}$, i.e.

$$\sum_{A, \alpha, \beta} \xi_{n+1 \ \alpha \beta} \omega_\beta \wedge \xi_{n+2 \ \alpha A} \omega_A = 0 \quad \text{or} \quad \sum_{A, \alpha, \beta} \xi_{n+1 \ \alpha \beta} \xi_{n+2 \ \alpha A} \omega_\beta \wedge \omega_A = 0.$$

Comparing the coefficients in the last equation, $\sum_\alpha \xi_{n+1 \ \alpha \beta} \xi_{n+2 \ \alpha i} = 0$ or

$$(12) \quad \sum_\beta \xi_{n+1 \ \alpha \beta} \xi_{n+2 \ i \beta} = 0.$$

However, our hypothesis implies that the q vectors $\Xi_\alpha = (\xi_{n+1 \ \alpha \ p+1}, \dots, \xi_{n+1 \ \alpha \ p+q})$ in R^q must be linearly independent. To prove this fact, let

$\sum_{\alpha} c_{\alpha} \Xi_{\alpha} = 0$ or $\sum_{\alpha} c_{\alpha} \xi_{n+1 \alpha \beta} = 0$, for any β , or $\sum_{\alpha} c_{\alpha} \omega_{n+1 \alpha} = 0$. By (3), this implies $\sum_{\alpha} c_{\alpha} T_{e_{\alpha}} y = \sum_{\alpha} c_{\alpha} \omega_{\alpha n+2}(y) f_2 + (0) f$, or

$$(13) \quad T_y \left(\sum_{\alpha} c_{\alpha} e_{\alpha} \right) \text{ is parallel to } f_2 \text{ for all } y.$$

Since $(a_{\alpha\beta})$ is nonsingular, we can find d_{α} such that $\sum_{\alpha} d_{\alpha} a_{\alpha\beta} = c_{\beta}$. Then

$$\begin{aligned} T_y \left(\sum_{\alpha} d_{\alpha} e'_{\alpha} \right) &= T_y \left(\text{a vector tangent to } U_1 + \sum_{\alpha} c_{\alpha} e_{\alpha} \right) \\ &= \text{a vector parallel to } f_2 + T_y \left(\sum_{\alpha} c_{\alpha} e_{\alpha} \right). \end{aligned}$$

But, by (6), the left-hand side is a vector parallel to f_1 . By (13), this is only possible if $T_y(\sum_{\alpha} d_{\alpha} e'_{\alpha}) = 0$ for all y ! Our hypothesis implies $\sum_{\alpha} d_{\alpha} e'_{\alpha} = 0$ or $d_{\alpha} = 0$ or finally $c_{\alpha} = 0$. Back to (12), as Ξ_{α} span E^q , (12) implies

$$(14) \quad \xi_{n+2 \ i \beta} = 0.$$

Now use $\bar{\nabla}_{e_A} f_2 = 0$, or $\sum_A a_{\alpha A} \bar{\nabla}_{e_A} f_2 = 0$ to get

$$\sum_A a_{\alpha A} \left[\sum_B \omega_{n+2 \ B}(e_A) e_B + \omega_{n+2 \ n+1}(e_A) f_1 \right] = 0.$$

In particular, $\sum_A a_{\alpha A} \omega_{n+2 \ B}(e_A) = 0$, or $\sum_A a_{\alpha A} \xi_{n+2 \ B \ A} = 0$. By (14), this reduces to $a_{\alpha i} \xi_{n+2 \ j i} = 0$. Again the linear independence of the p vectors $(\xi_{n+2 \ j 1}, \dots, \xi_{n+2 \ j p})$ in R^p (proof is the same as before) implies $a_{\alpha i} = 0$. This completes the proof of the following:

LEMMA A. *At each point $m \in M$, leaves in U_1 are orthogonal to leaves in U_2 .*

By Lemma 3.7 of [1], each leaf L_1 (resp. L_2) in U_1 (resp. U_2) belongs to a linear subvariety $[L_1]$ (resp. $[L_2]$) of dimension $p+1$ (resp. $q+1$), and L_i is the boundary of a convex set in $[L_i]$. Fixing a point $m \in M$, let $[L_1]_0, [L_2]_0$ be the linear subvarieties passing through m . Take the decomposition $R^{n+2} = [L_1]_0 \times [L_2]_0$ and let p_i be the orthogonal projections of R^{n+2} onto $[L_i]_0$. Let S be the set of points n of M such that the leaves through n are parallel to respectively $[L_1]_0$ and $[L_2]_0$. It is obvious that S is open and closed in M and hence equal to M . Define a map $g: M \rightarrow L_1 \times L_2$ by

$$g(n) = (p_1 \circ f(n), p_2 \circ f(n)).$$

For $n \in M$ let the leaves passing through n be L'_1 and L'_2 . $p_1 \circ f(n)$ ($p_2 \circ f(n)$) is the unique point of intersection of $[L'_2]$ and $[L_1]$ ($[L'_1]$ and $[L_2]$) which is the same as the intersection of L'_2 and L_1 (L'_1 and L_2), because two leaves from different families already intersect in a unique point [1, Lemmas 3.8

and 3.10]. This proves that g is well defined. Similarly, we prove that g is one-to-one and onto. Moreover, the differential of g is precisely our decomposition of the tangent space of M into two orthogonal distributions. Using the definition of the product of two Riemannian manifolds we state our main theorem:

THEOREM A. *Let $f: M^n \rightarrow R^{n+2}$ be a tight isometric immersion of a compact manifold M^n of nonnegative curvature. Assume $\mu(M)=4$ and $T_x x \neq 0$ for $x \neq 0$. Then M^n is isometric to the product of two convex hypersurfaces $f_1: L_1 \rightarrow R^p$ and $f_2: L_2 \rightarrow R^p$. Moreover, $f = (f_1, f_2)$.*

By Corollary 2 of [2], we may state a rigidity theorem.

COROLLARY. *Let M^n be a simply connected compact manifold of non-negative curvature. If M^n admits an immersion of the type described in the last theorem, then any isometric immersion of M^n in R^{n+2} is rigid.*

4. Tight isometric flat surfaces in R^4 . In the special case $n=2$, a result of [1] says that a compact surface of nonnegative curvature can be tightly, isometrically immersed into R^4 only if $\mu(M) \leq 4$ and $\mu(M)=4$ only if the surface is flat. To characterize tight flat surfaces in R^4 , we repeatedly use the hypothesis $T_x x \neq 0$ for $x \neq 0$ (i.e. no asymptotic direction) in the last section. In this section we will prove Theorem A without the extrinsic hypothesis $T_x x \neq 0$ for $x \neq 0$.

As to the intuitive ideas behind this paper, we note that tightness implies crowdedness of total curvature [1] and that this crowdedness of total curvature leads to an intrinsic way of splitting the manifold locally. For a surface in R^4 we observe that the points having asymptotic vectors make no contribution at all to the total curvature. Therefore, we still have crowdedness of total curvature on those areas without asymptotic vectors, and there we can apply the result of [1].

Let M^2 be a surface of flat metric with $\mu(M)=4$ and let $f: M^2 \rightarrow R^4$ be a tight isometric immersion. Write $M^2 = A \cup A' \cup A''$, where A is the set of points m such that $T_x x \neq 0$ for $x \neq 0$ at m ; and A' is the set of m such that $T \neq 0$ and there is a vector $x \neq 0$ with $T_x x = 0$; and A'' is the set of m where $T \equiv 0$. By the definition of tightness $\mu(M) = \tau(M)$, where

$$\tau(M) = \frac{1}{2\pi^2} \int_B |\det S_z| |w| = \frac{1}{2\pi^2} \int_{B'} |\det S_z| |w|$$

(B is the bundle of unit normals and B' is the restriction of B over A), since $\det S_z = 0$ for z normal to M on $A' \cup A''$. Our previous work [1] shows that A is foliated by two families of curves $\mathcal{U}$ and $\mathcal{V}$, where $\mathcal{U}$ (respectively $\mathcal{V}$) is the family of integral curves of the vector field U

(respectively V) defined uniquely by

$$(15) \quad \begin{aligned} x \in U & \text{ if and only if } T_x y \text{ parallel to } f_2 \text{ for all } y \\ (x \in V & \text{ if and only if } T_x y \text{ parallel to } f_1 \text{ for all } y), \end{aligned}$$

where f_1, f_2 give the normal frame of A . Most proofs in [1] can be applied to A , except we cannot infer that $\bar{\nabla}_x f_1 = 0$ for $x \in U$ (notice that we used U_1, U_2 for U, V in [1]), since U, V are one-dimensional.

But the following alternative way will fill up the gap. Take two unit vectors $e_1 \in U, e_2 \in V$. By (15), we have

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 &= \alpha e_1 + \beta e_2 + \gamma f_2, & \gamma &\neq 0, \\ \bar{\nabla}_{e_1} e_2 &= \alpha e_1 + \beta e_2. \end{aligned}$$

Hence

$$\begin{aligned} \bar{\nabla}_{e_1} \bar{\nabla}_{e_1} e_1 &= \alpha' e_1 + \beta' e_2 + \gamma' f_2 + \alpha(\alpha e_1 + \beta e_2 + \gamma f_2) \\ &\quad + \beta(\alpha e_1 + \beta e_2) + \gamma \bar{\nabla}_{e_1} f_2, \end{aligned}$$

where primes denote the differentiation along $\mathcal{U}$ curves. Since the hyperplane $f_1^\perp$ spanned by f_2, e_1, e_2 is a supporting hyperplane of M in R^4 (Lemma 3.9 of [1]) and both $e_1, \bar{\nabla}_{e_1} e_1$ belong to $f_1^\perp$, $\bar{\nabla}_{e_1} \bar{\nabla}_{e_1} e_1$ must also in $f_1^\perp$ [7, Corollary 3.4]. Thus

$$\langle \bar{\nabla}_{e_1} f_2, f_1 \rangle = 0 \quad \text{or} \quad \langle \bar{\nabla}_{e_1} f_1, f_2 \rangle = 0.$$

However, $\langle \bar{\nabla}_{e_1} f_1, f_1 \rangle = \frac{1}{2} e_1 \langle f_1, f_1 \rangle = 0$. So finally $\bar{\nabla}_{e_1} f_1 = 0$; that is, along each $\mathcal{U}$ curve, f_1 is parallel. In other words, the supporting hyperplane $f_1^\perp$ contains $\mathcal{U}$ curve. By Kuiper [6, Lemma 1], such a curve must be a plane convex arc. Moreover, all $\mathcal{U}$ curves are orthogonal to $\mathcal{V}$ curves (proved in the last section) and we have

LEMMA 1. *Each component of A is isometric to the product of two open convex 2-planar arcs.*

PROOF. Choose two mutually orthogonal geodesics in a component C_0 of A , each from one of $\mathcal{U}$ and $\mathcal{V}$, which are longest among all such curves in C_0 . Denote the curves respectively by $\alpha(s), 0 < s < a_1$, and $\beta(s), 0 < s < b_1$ (s denotes the arc-length) and let the 2-planes containing them be $[\alpha(0, a_1)]$ and $[\beta(0, b_1)]$ respectively. Let our euclidean space be decomposed into $[\alpha(0, a_1)] \times [\beta(0, b_1)]$. It suffices to prove that C_0 is the Riemannian product $\alpha(0, a_1) \times \beta(0, b_1)$ of the open convex arcs $\alpha(0, a_1)$ and $\beta(0, b_1)$. Explicitly we will show that the map $\psi: C_0 \rightarrow \alpha(0, a_1) \times \beta(0, b_1)$ which sends $m \in C_0$ to

$$\begin{aligned} &(\alpha(0, a_1) \cap V(m), \beta(0, b_1) \cap U(m)) \\ &= ([\alpha(0, a_1)] \cap [V(m)], [\beta(0, b_1)] \cap [U(m)]) \end{aligned}$$

is an isometry, where $U(m)$ and $V(m)$ are the leaves through m of $\mathcal{U}$ and $\mathcal{V}$ respectively. It is clear that ψ is well defined and one-to-one. Also ψ preserves Riemannian structure, because the maps $p_1: U(m) \rightarrow \alpha(0, a_1)$ and $p_2: V(m) \rightarrow \beta(0, b_1)$ defined by orthogonal projection are isometries. From now on, we will omit the isometry ψ and identify m with $\psi(m)$. To complete the proof, it remains to prove that ψ is onto. Suppose not; then pick $m = (m_1, m_2) \in \alpha(0, a_1) \times \beta(0, b_1)$ on the boundary of C_0 . By definition of A (an open set), there is an asymptotic direction at m . However, for $x \neq 0$ tangent to $U(m)$ (defined by taking limit of $U(m')$, $m' \in C_0$), we have $T_x x \neq 0$, since $p_1: (U(m))^- \rightarrow \alpha(0, a_1)$ is a local isometry. Similarly, for $x \neq 0$ tangent to $V(m)$, we have $T_x x \neq 0$. Therefore $T_x x \neq 0$, for all $x \neq 0$. This contradicts the existence of an asymptotic direction at m . Q.E.D.

In particular, the proof shows that the four edges of the rectangle belong to A' , and four corners of the rectangle belong to A'' . Moreover, along $(\alpha(0, a_1), \{\beta(0), \beta(b_1)\})$, we have $T_x x = 0$ for x tangent to $\beta(0, b_1)$, and along $(\{\alpha(0), \alpha(a_1)\}, \beta(0, b_1))$, $T_x x = 0$ for x tangent to $\alpha(0, a_1)$.

LEMMA 2. *Each component of A' adjacent to a component of A is isometric to the product of a closed line segment and an open arc which forms the border edge of the component of A .*

PROOF. In the terminology of [2], [9], A' is the set of points where the index of relative nullity of the immersion is one. Let C'_1 be the component of A' adjacent to C_0 of Lemma 1 and with $(\alpha(0, a_1), \beta(b_1))$ as border curve. It is well known (for example [9]) that C'_1 is foliated by two families of curves, one of them consisting of line segments only. Being the limit of the tangent lines to curves of $\mathcal{V}$ in C_0 , these lines must be parallel to each other. (This is the place where we do not use the completeness of A' which is essential in [9].) Since $(\alpha(0, a_1), \beta(b_1))$ is the curve of one of the families which does not consist of a line segment and $(\alpha(0, a_1), \beta(b_1))$ is a 2-planar curve, C'_1 actually lies within a hyperplane. Therefore C'_1 is a portion of a product. The first factor of the product is $\alpha(0, a_1)$, and the second factor of the product is the line segment tangent to $\beta(s)$ at $(\alpha(0, a_1), \beta(b_1))$. It remains to prove that by cutting a segment along the line, the set C'_1 agrees with the product set. This can be done as in the proof of Lemma 1. Q.E.D.

Notice that the opposite edge of $(\alpha(0, a_1), \beta(b_1))$ in C'_1 (a rectangle) must meet another rectangle in A . Let the three consecutive rectangles be C_0, C'_1, C_2 . The 2-plane of the border edge between C_0 and C'_1 is parallel to the one between C'_1 and C_2 , since they are parallel to 2-planes of leaves in C'_1 (lies within a hyperplane). Moreover, if we hook up the leaves across the border edges, we get a curve across C_0, C'_1, C_2 which lies

within a 2-plane orthogonal to the 2-plane of the edge $(\alpha(0, a_1), \beta(b_1))$. This process will give a band B formed by a chain of rectangles $C_0, C'_1, C_2, C'_3, \dots$ (may be infinite in number). At the same time we extend the curve $\beta(s), 0 \leq s \leq b_1$ to get a curve $\beta(s), 0 \leq s \leq b$ across this chain of rectangles which lies in a 2-plane orthogonal to the 2-plane of the edge $(\alpha(0, a_1), \beta(b_1))$.

LEMMA 3. *The band B is isometric to the product of an open convex plane arc $\alpha(0, a_1)$ and a closed convex plane curve $\beta(0, b)$.*

PROOF. It remains to show that the curves $\beta(0, b)$ on the band are convex plane curves. Since $\beta(0, b)$ is the intersection of M^2 with a supporting hyperplane of Lemma 3.9 of [1], they are all Top sets [5], [6]. Since M^2 is tight, all Top sets are convex plane curves (Lemma 1 of [6]).

LEMMA 4. *The open convex arc $\alpha(0, a_1)$ in Lemma 3 can be extended to a closed convex plane curve $\alpha(0, a)$ so that the product band B extends to all of M^2 .*

PROOF. As before, let B be covered by rectangles $C_0, C'_1, C_2, C'_3, \dots$. Start with C_0 . Instead of crossing the border curve $(\alpha(0, a_1), \beta(b_1))$ as in the proof of Lemma 2, we cross $(\alpha(a_1), \beta(0, b_1))$ to get a rectangle D'_0 in A' . By Lemma 2 $\alpha(0, a_1)$ extends across $(\alpha(a_1), \beta(0, b_1))$ so that D'_0 is the product of the extended portion $\alpha(a_1, a_2)$ of $\alpha(0, a_1)$ and those portions $\beta(0, b_1)$ of $\beta(0, b)$ which lie in C_0 . Meanwhile, from C'_1 we cross $(\alpha(a_1), \beta(b_1, b_2))$ to get a plane D''_1 of A'' , also from C_2 to D' and so on. It can be seen that B will be doubled and tripled and so on, and M will be covered by the product $\alpha(0, a) \times \beta(0, b)$. By the reason quoted from [6] in Lemma 3, $\alpha(0, a)$ is a closed convex plane curve. Q.E.D.

We state final result as the theorem promised in §1.

THEOREM. *Let $f: M^2 \rightarrow R^4$ be a tight isometric immersion of a compact flat manifold. Then f is a product embedding of plane convex closed curves (hence in particular M is a flat torus).*

REMARKS. (1) Tightness is necessary, because any product of plane curves will give an isometric immersion of the flat torus in R^4 which is not necessarily tight.

(2) Flatness is also necessary, because tightness is preserved under any affine transformation; however, the Riemannian structure is not.

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