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METRIC INEQUALITIES AND THE ZONOID PROBLEM

H. S. WITSENHAUSEN

ABSTRACT. For normed spaces the hypermetric and quasihypermetric properties are equivalent and imply the quadrilateral property. The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric. The unit ball of l_p^n is not a zonoid for $n=3$, $p<\log 3/\log 2$, and for $p\leqq 2-(2n\log 2)^{-1}+o(n^{-1})$. The elliptic spaces $\mathcal{E}^d$, $d>1$, are not quasihypermetric.

A metric space (S, d) is said to be *hypermetric* (Kelly [3]) when

$$\sum_{i,j=1}^{n} w_i w_j d(x_i, x_j) \leqq 0 \tag{1}$$

for all $n>0$, $x_1, \cdots, x_n$ in S, and $w_1, \cdots, w_n$ integers with sum 1. This implies [5] that (1) also holds for real w_i of sum 0, which is called the *quasihypermetric* property.

A *piecewise linear inequality* (PLI) is a relation of the form

$$\sum_{i=1}^{k} c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| \geqq 0 \tag{2}$$

which holds for all n-tuples $x_1, \cdots, x_n$ of real numbers, with fixed real c_i and a_{ij}. An example is the *quadrilateral* inequality [8]

$$|x| + |y| + |z| - |x+y| - |y+z| - |z+x| + |x+y+z| \geqq 0. \tag{3}$$

Since the real line is hypermetric [4], (1) generates an infinite family of PLI's of the form

$$\sum_{i,j=1}^{n} (-w_i w_j)\, |x_i - x_j| \geqq 0$$

for w_i integers of sum 1 and for w_i reals of sum 0.

The PLI (2) is said to *extend* to the normed space N if it holds with the absolute value function replaced by the norm and $x_1, \cdots, x_n$ arbitrary elements of N.

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A *zonoid* [**1**], [**2**] is a convex body belonging to the closure (in the Hausdorff set metric) of the class of zonotopes (polytopes which are Minkowski sums of segments).

The theorems of I. J. Schoenberg [**7**] and P. Lévy [**6**] imply that the above concepts are related.

PROPOSITION 1. *For a real normed space N the following* 3 *properties are equivalent.*

(i) *every PLI extends to N*,

(ii) *N is quasihypermetric*,

(iii) $e^{-\|x\|}$ *is positive definite on N.*

PROOF. If every PLI extends to N then in particular N is quasihypermetric. Following an argument of Schoenberg, consider $n+1$ points $x_0, \cdots, x_n$ in N with weights $-\sum_{i=1}^n w_i, w_1, \cdots, w_n$ where the w_i are arbitrary reals. This yields

$$\sum_{i,j=1}^{n} w_i w_j(\|x_i - x_0\| + \|x_j - x_0\| - \|x_i - x_j\|) \geqq 0,$$

that is, the parenthesis is positive definite. Then its exponential is positive definite and, absorbing $e^{\|x_i - x_0\|}$ into w_i, $e^{-\|x\|}$ is shown to be positive definite. Conversely, if $e^{-\|x\|}$ is positive definite on N it is positive definite on every finite dimensional subspace of N. By Lévy's theorem [**6**], [**1**] these subspaces are isometrically isomorphic to subspaces of $L_1(0, 1)$ to which any PLI extends by integration. Since each PLI involves only finite systems of vectors, it extends to all of N.

In [**4**] Kelly raised the question of the possible relations between the hypermetric and quadrilateral properties in normed spaces. Applying Proposition 1 one has

COROLLARY 1.1. *For real normed spaces, the hypermetric and quasihypermetric properties are equivalent and they imply the quadrilateral property.*

For $1 \leqq p \leqq 2$, $e^{-\|x\|}$ is known [**7**] to be positive definite on $L_p(0, 1)$, hence

COROLLARY 1.2. $L_p(0, 1)$ (*and a fortiori* l_p^n) *is hypermetric and quadrilateral for* $1 \leqq p \leqq 2$.

This had been conjectured by Kelly [**4**] and the Smileys [**8**]. For finite dimensional real normed spaces (Minkowski spaces) the positive definiteness of $e^{-\|x\|}$ is equivalent [**1**] to the property that the unit ball of the dual space is a zonoid. Thus one has

COROLLARY 1.3. *The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric.*

Thus the known fact that all Minkowski planes are hypermetric **[4]** follows from the elementary fact that all centrally symmetric convex polygons are sums of segments.

For $n \geqq 3$, let p_n be the smallest p such that the unit ball of l_p^n is a zonoid. One has $p_3 \leqq p_n \leqq p_{n+1} \leqq 2$, Bolker **[1]**, **[2]** has conjectured that $p_3 = 2$. He reports the following bounds of Rosenthal: $p_3 > \log 9/\log 7$ and $p_n > 2 \log n/\log 3n$, hence $p_n \geqq 2 - \log 9/\log n + o((\log n)^{-1})$.[1] These bounds can be substantially improved.

PROPOSITION 2. *One has* $p_3 \geqq \log 3/\log 2$ *and* $p_n \geqq 2 - 1/2n \log 2 + o(n^{-1})$.

PROOF. For $n=3$, $p < \log 3/\log 2$ the quadrilateral inequality in the dual space is violated for $x=(1, 1, -1)$, $y=(1, -1, 1)$, $z=(-1, 1, 1)$, as observed by the Smileys **[8]**.[2] For large even $n=2m$, consider the quasihypermetric inequality in the dual l_q^{2m}, with $w_i = 1$ at the 2^m points with the first m coordinates equal to ± 1 and the last m coordinates 0, $w_i = -1$ at the 2^m points with first m coordinates 0 and the last m equal to ± 1. All distances between the two sets are $(2m)^{1/q}$ while distances within each set are of the form $2k^{1/q}$ with $0 \leqq k \leqq m$. Counting the number of occurrences of each distance, a violation of the inequality is seen to require

$$2\left(2^{m-1} \sum_{k=0}^{m} \binom{m}{k} (2k^{1/q})\right) > 2^{2m}(2m)^{1/q}$$

or $2E\{k^{1/q}\} > (2m)^{1/q}$ with k binomially distributed. For large m, expand $k^{1/q}$ about the mean $k=m/2$ and let $1/q = \frac{1}{2} + \varepsilon$. Then the violation occurs for $\varepsilon < -(16m \log 2)^{-1} + o(m^{-1})$, so that $p_n \geqq 2 - (2n \log 2)^{-1} + o(n^{-1})$ as claimed.

Kelly **[3]** has shown that spherical spaces are hypermetric. This no longer holds when antipodes are identified.

PROPOSITION 3. *The elliptic plane* $\mathscr{E}^2$ *is not quasihypermetric.*

PROOF. Assume the opposite, and consider the function, defined for μ in $C(\mathscr{E}^2)^*$, by $F(\mu) = \int \mu(dx) \int \mu(dy)\overline{xy}$, where $\overline{xy}$ is the elliptic distance and the integrals range over the compact space $\mathscr{E}^2$. By (1) the function is nonpositive, hence concave on the subspace $\{\mu | \int \mu(dx) = 0\}$. The concavity holds as well on the parallel subspace $\{\mu | \int \mu(dx) = 1\}$ and in particular on the set $\mathscr{P}$ of probability measures on $\mathscr{E}^2$. For μ in $\mathscr{P}$ and τ in the compact group G of isometries of $\mathscr{E}^2$, let μ^* be the mixture of the displaced measures $\mu \circ \tau$ under normalized Haar measure on G. Then μ^* is the uniform

[1] Thus $L_p(0, 1)$ is not hypermetric for $p > 2$.

[2] Alternatively, the hypermetric inequality is violated for the choice of $w_i = 1$ at $(\pm 1, \pm 1, 0)$ and $w_i = -1$ at $(0, 0, 0)$, $(0, 0, \pm 1)$.

distribution on $\mathscr{E}^2$ and by concavity $F(\mu^*) \geqq F(\mu)$. However, the distribution μ assigning equal probabilities to the vertices of an equilateral triangle of side length D, the diameter of $\mathscr{E}^2$, yields $F(\mu)=2D/3$ while $F(\mu^*)=2D/\pi$, a contradiction.

That $\mathscr{E}^2$ is not hypermetric already follows from the violation of the hypermetric inequality that occurs for the choice of $w_i=-1$ at 3 mutually orthogonal lines and $w_i=+1$ at their 4 trisectors.

Since $\mathscr{E}^2 \subset \mathscr{E}^d$ for $d>2$ one has

COROLLARY 3.1. *For $d>1$ the elliptic space $\mathscr{E}^d$ is not quasihypermetric.*

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