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A REGULAR DETERMINANT OF BINOMIAL COEFFICIENTS

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ABSTRACT. Let n be a positive integer and suppose that each of $\{a_p\}_1^n$ and $\{c_p\}_1^n$ is an increasing sequence of nonnegative integers. Let M be the $n \times n$ matrix such that $M_{ij} = C(a_i, c_j)$, where $C(m, n)$ is the number of combinations of m objects taken n at a time. We give an explicit formula for the determinant of M as a sum of nonnegative quantities. Further, if $a_i \geqq c_i$, $i = 1, 2, \cdots, n$, we show that the determinant of M is positive.

A formula for the determinant of binomial coefficients

$$\begin{vmatrix} \binom{a_1}{c_1} & \binom{a_1}{c_2} & \binom{a_1}{c_3} & \cdots & \binom{a_1}{c_n} \\ \binom{a_2}{c_1} & \binom{a_2}{c_2} & \binom{a_2}{c_3} & \cdots & \binom{a_2}{c_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{a_n}{c_1} & \binom{a_n}{c_2} & \binom{a_n}{c_3} & \cdots & \binom{a_n}{c_n} \end{vmatrix}$$

$(a_i < a_{i+1};\ c_i < c_{i+1})$ as a sum of nonnegative quantities is presented in this paper. Thus the above determinant is nonnegative. Moreover, the determinant is positive if no diagonal entry is zero.

Many special cases of this type of determinant have been studied, principally last century. Sir Thomas Muir, in his five volumes on determinants, cites and summarizes several (see [**3**, vol. III, pp. 447–462]). Muir points out that the determinant above is the object of a paper in 1879 by S. Günther [**1**]. Of this paper, Muir [**3**, Vol. III, p. 462] says, "The outcome is not very satisfying."

More recently John Neuberger has studied some special cases of this (see [**4**, Lemmas 1, 2, and 3]).

THEOREM 1. *Suppose that n is a positive integer and each of $\{a_p\}_1^n$ and $\{c_p\}_1^n$ is an increasing sequence of nonnegative integers. Let M be*

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the $n\times n$ matrix such that

$$M_{pq} = \binom{a_p}{c_q} \qquad (p, q = 1, 2, \cdots, n).$$

The determinant of M is

$$\frac{\prod_{k=1}^{n} \binom{a_k}{c_1}}{\prod_{k=2}^{n} \binom{c_k}{c_1} \prod_{j=3}^{n} \prod_{k=j}^{n} \binom{c_k - c_{j-2} - 1}{c_{j-1} - c_{j-2} - 1}}$$

$$\times \sum_{p \text{ in } T(n,a)} \prod_{j=1}^{n-1} \prod_{k=1}^{n-j} \binom{a_k + \sum_{\alpha=1}^{j} p_{\alpha k} - c_j - 1}{c_{j+1} - c_j - 1},$$

where $T(n, a)$ denotes the collection of all $n\times n$ matrices p having the property that if each of q and i is an integer in $[1, n]$ then $p_{qi}=0$ if $q+i>n$ and, if $q+i\leqq n$, then p_{qi} is an integer and

$$1 \leqq p_{qi} \leqq a_{i+1} - a_i + \sum_{\alpha=1}^{q-1} (p_{\alpha,i+1} - p_{\alpha i}).$$

The sum $\sum_{p \text{ in } T(n,a)}$ in the formula is actually a multiple sum involving $(n-1)!$ $\sum$'s. For example, if n is 4 the sum is

$$\sum_{p_{11}=1}^{a_2-a_1} \sum_{p_{12}=1}^{a_3-a_2} \sum_{p_{13}=1}^{a_4-a_3} \sum_{p_{21}=1}^{a_2-a_1+p_{12}-p_{11}} \sum_{p_{22}=1}^{a_3-a_2+p_{13}-p_{12}} \sum_{p_{31}=1}^{a_2-a_1+p_{12}-p_{11}+p_{22}-p_{21}}$$

THEOREM 2. *In addition to the suppositions of Theorem 1 let us suppose that*

$$\binom{a_k}{c_k} \neq 0 \qquad (k = 1, 2, \cdots, n).$$

Then the determinant of M is positive, so that the rows and columns of M are linearly independent sets of n-tuples.

If the integer n of Theorem 1 is 1, the formula gives $\binom{a_1}{c_1}$. We will prove Theorem 1 by induction. We need the following formulas. (These formulas also provide a proof of Theorem 1 if n is 2.)

Suppose that each of a, b, i, and j is a nonnegative integer and $a<b$ and $i<j$. Then

$$\text{(1)} \qquad \binom{a}{j} = \binom{a}{i}\binom{a-i}{j-i}\bigg/\binom{j}{i}$$

and

$$\binom{b}{i} - \binom{a}{i} = \sum_{s=1}^{b-a} \binom{a+s-1}{i-1}. \tag{2}$$

In the setting of Theorem 1, det M is denoted $D(n, a, c)$, depending on n and the sequences a and c.

We do quite a bit of summation over sets below. The following formulas are applied.

I. If S is a finite set and ϕ is a function reversible (one-to-one) on S and f is a function from $\phi(S)$ to the numbers then

$$\sum_{s \text{ in } \phi(S)} f(s) = \sum_{s \text{ in } S} f(\phi(s)).$$

II. If A is a partition of the finite set S and f is a function from S to the numbers then

$$\sum_{s \text{ in } S} f(s) = \sum_{\alpha \text{ in } A} \sum_{s \text{ in } \alpha} f(s).$$

PROOF OF THEOREM 1. Suppose that n is an integer, $n>1$, and "Theorem 1 is true for n and for $n-1$". Suppose that each of $a=\{a_p\}_1^{n+1}$ and $c=\{c_p\}_1^{n+1}$ is an increasing sequence of nonnegative integers. We wish to calculate $D(n+1, a, c)$, the determinant of the $(n+1)\times(n+1)$ matrix M such that

$$M_{pq} = \binom{a_p}{c_q} \qquad (p, q = 1, 2, \cdots, n+1).$$

If v is a sequence and r is an integer then $R(v, r)$ is the sequence such that $R(v, r)_p = v_p$ if $p<r$ and $R(v, r)_p = v_{p+1}$ if $p \geqq r$.

Please note that $D(n, R(a, r), c)$ is the determinant of the matrix M after deleting the rth row and the last column. By our inductive hypothesis our formula, given in Theorem 1, applies to $D(n, R(a, r), c)$, $r=1, 2, \cdots, n+1$.

If N is a positive integer, A is a sequence, C is a sequence, and p is an $N\times N$ matrix of numbers then

$$P(N, A, C, p) = \prod_{j=1}^{N-1} \prod_{k=1}^{N-j} \binom{A_k + \sum_{\alpha=1}^{j} p_{\alpha k} - C_j - 1}{C_{j+1} - C_j - 1},$$

$$D(N, C) = \prod_{k=2}^{N} \binom{C_k}{C_1} \prod_{j=3}^{N} \prod_{k=j}^{N} \binom{C_k - C_{j-2} - 1}{C_{j-1} - C_{j-2} - 1},$$

and

$$K(N, A, C) = \left\{ \prod_{k=1}^{N} \binom{A_k}{C_1} \right\} \Big/ D(N, C).$$

Consequently, the formula for the determinant in Theorem 1 is "simply"

$$K(n, a, c) \sum_{p \text{ in } T(n,a)} P(n, a, c, p).$$

Now, expanding by the last column of M,

$$\det M = D(n+1, a, c) = \sum_{r=1}^{n+1} (-1)^{r+n+1} \binom{a_r}{c_{n+1}} D(n, R(a, r), c)$$

$$= \sum_{r=1}^{n+1} (-1)^{r+n+1} \binom{a_r}{c_{n+1}} K(n, R(a, r), c) \sum_{p \text{ in } T(n, R(a,r))} P(n, R(a, r), c, p)$$

$$= \left[\binom{a_{n+1}}{c_1} \Big/ \binom{c_{n+1}}{c_1}\right] K(n, a, c) \sum_{r=1}^{n+1} (-1)^{r+n+1} \binom{a_r - c_1}{c_{n+1} - c_1} \sum_{p \text{ in } T(n, R(a,r))} P(n, R(a, r), c, p)$$

(by formula (1))

$$= \sum_{r=1}^{n+1} f_r \sum_{p \text{ in } T(n, R(a,r))} P(n, R(a, r), c, p),$$

where

$$\mathscr{C} = (-1)^n K(n, a, c) \binom{a_{n+1}}{c_1} \Big/ \binom{c_{n+1}}{c_1} \quad \text{and} \quad f_r = \mathscr{C}(-1)^{r+1} \binom{a_r - c_1}{c_{n+1} - c_1}.$$

Here we need more notation. For each positive integer r, $\delta(r)$ is the sequence such that $\delta(r)_k = a_{k+1} - a_k$ if $k < r$ and $\delta(r)_k = a_{k+2} - a_{k+1}$ if $k > r$. Δa denotes $\delta(n+1)$. For $r = 1, 2, \cdots, n$, T_r is the set of all $n \times n$ matrices p such that if each of q and i is an integer in $[1, n]$ then $p_{qi} = 0$ if $q + i > n$ and, if $q + i \leqq n$ then p_{qi} is an integer, $1 \leqq p_{1i} \leqq \delta(r)_i$ and, if $q > 1$, $1 \leqq p_{qi} \leqq R(a, r)_{i+1} - R(a, r)_i + \sum_{\alpha=1}^{q-1} (p_{\alpha, i+1} - p_{\alpha i})$.

Now, continuing calculation,

$$\det M = f_1 \sum_{p \text{ in } T(n, R(a,1))} P(n, R(a, 1), c, p)$$

$$+ f_{n+1} \sum_{p \text{ in } T(n, R(a, n+1))} P(n, R(a, n+1), c, p)$$

$$+ \sum_{r=2}^{n} f_r \sum_{\substack{p \text{ in } T(n, R(a,r)): \\ p_{1, r-1} \leqq \delta(r)_{r-1}}} P(n, R(a, r), c, p)$$

$$+ \sum_{r=2}^{n} f_r \sum_{\substack{p \text{ in } T(n, R(a,r)): \\ p_{1, r-1} > \delta(r)_{r-1}}} P(n, R(a, r), c, p)$$

$$= \sum_{r=1}^{n} f_r \sum_{p \text{ in } T_r} P(n, R(a, r), c, p) + f_{n+1} \sum_{p \text{ in } T_n} P(n, R(a, n), c, p)$$

$$+ \sum_{r=2}^{n} f_r \sum_{p' \text{ in } T_{r-1}} P(n, R(a, r-1), c, p')$$

(applying sum formula I)

$$= \mathscr{C} \sum_{r=1}^{n} (-1)^r \left[\binom{a_{r+1} - c_1}{c_{n+1} - c_1} - \binom{a_r - c_1}{c_{n+1} - c} \right] \sum_{p \text{ in } T_r} P(n, R(a, r), c, p)$$

$$= \mathscr{C} \sum_{r=1}^{n} (-1)^r \sum_{s=1}^{a_{r+1}-a_r} \binom{a_r - c_1 + s - 1}{c_{n+1} - c_1 - 1} \sum_{p \text{ in } T_r} P(n, R(a, r), c, p)$$

(by formula (2)).

Here for the last time, we pause for more notation. If N is a positive integer and $d=\{d_k\}_1^N$ is a sequence of positive integers, then $V(N, d)$ is the collection of all sequences $v=\{v_k\}_1^N$ of integers having the property that if k is an integer in $[1, N]$ then $1 \leqq v_k \leqq d_k$. If $v=\{v_k\}_1^n$ is a sequence then $S(v)_k = v_k - c_1 - 1$ and $\gamma_k = c_{k+1} - c_1 - 1$, $k=1, 2, \cdots, n$. So, now,

$$\det M = \mathscr{C} \sum_{r=1}^{n} (-1)^r \sum_{s=1}^{a_{r+1}-a_r} \binom{S(a)_r + s}{\gamma_n} \sum_{v \text{ in } V(n-1, \delta(r))} \sum_{p' \text{ in } T(n-1, R(a,r)+v)}$$

$$\cdot \prod_{k=1}^{n-1} \binom{S(R(a, r) + v)_k}{\gamma_1} P(n - 1, R(a, r) + v, \gamma, p')$$

(by the sum formulas)

$$= \mathscr{C} \sum_{r=1}^{n} (-1)^r \sum_{v \text{ in } V(n, \Delta a)} \binom{S(a)_r + v_r}{\gamma_n} \sum_{p \text{ in } T(n-1, R(a+v, r))}$$

$$\cdot \prod_{k=1}^{n-1} \binom{S(R(a + v, r))_k}{\gamma_1} P(n - 1, S(R(a + v, r)), \gamma, p)$$

$$= C \sum_{v \text{ in } V(n, \Delta a)} \sum_{r=1}^{n} (-1)^{r+n} \binom{S(a + v)_r}{\gamma_n} K(n - 1, R(S(a + v), r), \gamma)$$

$$\cdot \sum_{p \text{ in } T(n-1, R(S(a+v), r))} P(n - 1, R(S(a + v), r), \gamma, p),$$

where C denotes $\prod_{k=1}^{n+1} \binom{a_k}{c_1} / \prod_{k=2}^{n+1} \binom{c_k}{c_1}$.

Using calculations similar to those way back when we began to calculate det M—just the first two lines—and recalling that the "inductive hypothesis holds for $n-1$", we see that, for v in $V(n, \Delta a)$, $D(n, S(a+v), \gamma)$ is precisely that number which begins $\sum_{r=1}^{n}$ above.

This allows us to use our inductive hypothesis for the determinants $D(n, S(a+v), \gamma)$ and conclude calculation of det M.

$$\det M = C \sum_{v \text{ in } V(n, \Delta a)} D(n, S(a + v), \gamma)$$

$$= C \sum_{v \text{ in } V(n, \Delta a)} K(n, S(a + v), \gamma) \sum_{p \text{ in } T(n, S(a+v))} P(n, S(a + v), \gamma, p)$$

$$= K(n + 1, a, c) \sum_{p \text{ in } T(n+1, a)} P(n + 1, a, c, p).$$

(This last is by application of the two summation formulas.) Now we have finished, for this last expression is just what our determinant formula yields for $n+1$.

PROOF OF THEOREM 2. Theorem 1 expresses det M as the product of a nonnegative quantity with a sum of nonnegative quantities. If k is an integer in $[1, n]$, $\binom{a_k}{c_k}>0$ and $a_k \geqq c_k$.

$$\prod_{k=1}^{n} \binom{a_k}{c_1} \geqq \prod_{k=1}^{n} \binom{a_1}{c_1} > 0,$$

and the leading coefficient is positive. Now we need to find a positive summand. Inductively we define p such that if each of q and i is an integer in $[1, n]$ then $p_{qi}=0$ if $q+i>n$ and, if $q+i \leqq n$,

$$p_{qi} = a_{i+1} - a_i + \sum_{\alpha=1}^{q-1} (p_{\alpha,i+1} - p_{\alpha i}).$$

Suppose that each of j and k is a positive integer and $k \leqq n-j$.

LEMMA. *If m is a nonnegative integer and $m \leqq j$ then*

$$a_k + \sum_{\alpha=1}^{j} p_{\alpha k} = a_{k+m} + \sum_{\alpha=1}^{j-m} p_{\alpha,k+m}.$$

PROOF. This is so if $m=0$. Let m be a nonnegative integer such that $m<j$ and the equation above is true. Then

$$\begin{aligned} a_{k+m} + \sum_{\alpha=1}^{j-m} p_{\alpha,k+m} &= a_{k+m} + p_{j-m,k+m} + \sum_{\alpha=1}^{j-m-1} p_{\alpha,k+m} \\ &= a_{k+m+1} + \sum_{\alpha=1}^{j-(m+1)} p_{\alpha,k+m+1}. \end{aligned}$$

Thus the lemma is true.

By the lemma, $a_k + \sum_{\alpha=1}^{j} p_{\alpha k} = a_{k+j}$. Thus

$$a_k + \sum_{\alpha=1}^{j} p_{\alpha k} - c_j - 1 = a_{k+j} - c_j - 1 \geqq a_{j+1} - c_j - 1 \geqq c_{j+1} - c_j - 1$$

so that

$$\binom{a_k + \sum_{\alpha=1}^{j} p_{\alpha k} - c_j - 1}{c_{j+1} - c_j - 1} > 0,$$

and there is a positive summand.

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