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FINITE DIMENSIONAL GROUP RINGS[1]

RALPH W. WILKERSON

ABSTRACT. A ring is right finite dimensional if it contains no infinite direct sum of right ideals. We prove that if a group G is finite, free abelian, or finitely generated abelian, then a ring R is right finite dimensional if and only if the group ring RG is right finite dimensional. A ring R is a self-injective cogenerator ring if R_R is injective and R_R is a cogenerator in the category of unital right R-modules; this means that each right unital R-module can be embedded in a direct product of copies of R. Let G be a finite group where the order of G is a unit in R. Then the group ring RG is a self-injective cogenerator ring if and only if R is a self-injective cogenerator ring. Additional applications are given.

1. **Introduction.** Let R always denote an associative ring with 1 and G a group with order $|G|$. The *group ring* of a group G and a ring R is the ring of all formal sums $\sum_{g\in G} r(g)g$ with $r(g) \in R$ and with only finitely many nonzero $r(g)$ [7]. For a right finite dimensional ring R, there exists an integer n such that R contains a direct sum of n-summands and the number of summands of any other direct sum in R is at most n. In this case, we write dim $R=n$. The ring R will be considered as a right R-module R_R and by finite dimensional we shall mean right finite dimensional.

It is known that if H is any semigroup with 1, then RH is a ring. In particular, the polynomial ring is a special case of this construction. Shock has shown that the right finite dimensional property carries over to polynomial rings [10]. This paper extends this result to group rings.

If R is a subring of Q and the identity of R is also the identity of Q, then R is a *right order* in Q if

(a) every nonzero divisor of R is a unit in Q, and

(b) every element of Q can be written in the form of cd^{-1} where c and d are in R and d is a nonzero divisor of R. We prove that if G is a finite group, then R is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of G is a zero-divisor in R

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if and only if RG is a right order in a self-injective cogenerator ring. Let G be a free abelian group. If R is a right order in a right Artinian ring then RG is a right order in a right Artinian ring.

2. **Finite dimensional group rings.**

It is always true that if RG is finite dimensional then R is finite dimensional; however, the converse is not in general true.

EXAMPLE 2.1. There exists a finite dimensional ring R and a group G such that the group ring RG is not finite dimensional. Let R be a field of characteristic zero and $G=\bigoplus \sum_{C_p}$ (for all prime p), where C_p is a cyclic group of order p. Then RG is not finite dimensional. This follows from the fact that RG is regular and the right ideal $\omega(C_p)$ of RG generated by $\{1-h \mid h \in C_p\}$ is principal [2]. So the question naturally arises as to when the group ring RG is finite dimensional.

PROPOSITION 2.2 (SHOCK [10]). *A ring R is finite dimensional if and only if the polynomial ring $R[x_1, x_2, \cdots]$ is finite dimensional. Furthermore,* $\dim R = \dim R[x_1, x_2, \cdots]$.

PROOF. See Theorem 2.6 of [10].

Let R be a subring of S, then we call S *a ring of right quotients of* R, if for every $0 \neq s \in S$ and for every $s' \in S$, there exists $r \in R$ such that $sr \neq 0$ and $s'r \in R$. Let $Q(R)$ denote the complete ring of quotients of R. It is well known that R is finite dimensional if and only if $Q(R)$ is, and in this case $\dim R = \dim Q(R)$. It is also known that if S is a ring of right quotients of R then $Q(R)$ is the complete ring of quotients of S [4].

THEOREM 2.3. *Let G be an infinite cyclic group, then R is finite dimensional if and only if RG is finite dimensional. Furthermore,* $\dim R = \dim RG$.

PROOF. Let S be a multiplicative semigroup isomorphic to the nonnegative integers. Then S is a semigroup with identity and is generated by the nonnegative powers of some element, say g. By Proposition 2.2, it is clear that RS is finite dimensional, since RS is just a polynomial ring in the variable g. Now S can be embedded in an infinite cyclic group G, which is generated by all powers of g. We need only show that RG is a ring of right quotients of RS. Let $r_1, r_2 \in RG$ with

$$\begin{aligned} 0 \neq r_1 &= r_1(g_1)g_1 + \cdots + r_1(g_n)g_n \\ &= r_1(g_1)g^{a_1} + \cdots + r_1(g_n)g^{a_n} \end{aligned}$$

and

$$\begin{aligned} r_2 &= r_2(h_1)h_1 + \cdots + r_2(h_m)h_m \\ &= r_2(h_1)g^{b_1} + \cdots + r_2(h_m)g^{b_m}. \end{aligned}$$

Let $k=\max\{|a_i|, |b_j|\}$ for all $1 \leqq i \leqq n$ and $1 \leqq j \leqq m$. It is clear that $r=g^k \in RS$, $r_1 r \neq 0$, and $r_2 r \in RS$. Hence, RG is finite dimensional. Also, $\dim Q(RS)=\dim RS=\dim R$ shows that $\dim R=\dim RG$. The converse is clear.

A *free abelian group* is a group which is a direct sum of infinite cyclic groups.

COROLLARY 2.4. *Let G be a free abelian group, then R is finite dimensional if and only if RG is finite dimensional. Furthermore*, $\dim R=\dim RG$.

PROOF. Let $H=S_1 \oplus S_2 \oplus \cdots$ where each S_i is a multiplicative semigroup isomorphic to the nonnegative integers. If R is finite dimensional then RH is finite dimensional by Proposition 2.2. Let $G=G_1 \oplus G_2 \oplus \cdots$, where S_i is embedded in the infinite cyclic group G_i, and now show that RG is a ring of right quotients of RH. The details are omitted. The converse and $\dim R=\dim RG$ follow easily.

LEMMA 2.5. *For a finite group G, the group ring RG is finite dimensional if and only if the ring R is finite dimensional. Also*, $\dim R \leqq \dim RG \leqq \dim R \cdot |G|$.

PROOF. Let G be finite, then RG_R is R-isomorphic to a direct sum of $|G|$ copies of the finite dimensional R-module R. Hence, RG is a finite dimensional R-module and therefore a finite dimensional RG-module. The converse and inequalities are clear.

THEOREM 2.6. *Let G be a finitely generated abelian group, then R is finite dimensional if and only if RG is finite dimensional. If H is the torsion subgroup of G, then* $\dim R \leqq \dim RG \leqq \dim R \cdot |H|$.

PROOF. If G is a finitely generated abelian group then $G \cong G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus H$ where $|H|<\infty$ and G_i for $1 \leqq i \leqq n$ is an infinite cyclic group. As in [**2**, p. 673], we define $A_1=RG_1$, $A_2=A_1G_2, \cdots, A_n=A_{n-1}G_n$, and $A=A_nH$; clearly $RG \cong A$. By Corollary 2.4 and Lemma 2.5, we see by induction that A is finite dimensional and consequently RG is finite dimensional. The converse and inequalities follow easily.

3. Applications.

Let $Z(R)$ denote the *right singular ideal* of R (**4**).

LEMMA 3.1. *Let G be a free abelian group, then $Z(RG)=Z(R)G$.*

PROOF. The proof uses the same technique as the proof of Theorem 2.7 of [**10**].

PROPOSITION 3.2 (CONNELL, [**2**]). *The group ring RG is semiprime if and only if R is semiprime and the order of no finite normal subgroup is a zero-divisor in R.*

PROOF. See the appendix of [**4**].

It is well known that a semiprime Goldie ring is a semiprime, finite dimensional ring with zero singular ideal.

COROLLARY 3.3. *Let G be a free abelian group. A ring R is a semiprime Goldie ring if and only if RG is a semiprime Goldie ring.*

PROOF. The proof is immediate.

PROPOSITION 3.4 (BURGESS, [**1**]). *If $Z(RG)=0$, then $Z(R)=0$ and the order of every finite normal subgroup of G is a nonzero-divisor in R.*

PROOF. See Theorem 4.8 of [**1**].

A *locally normal group* is one in which every finite subset is contained in a finite normal subgroup.

PROPOSITION 3.5 (BURGESS, [**1**]). *Assume that G is locally normal and the order of every finite normal subgroup of G is a nonzero-divisor in R. If $Z(R)=0$, then $Z(RG)=0$.*

PROOF. See 4.9 of [**1**].

COROLLARY 3.6. *Let G be a finitely generated abelian group. Then R is a semiprime Goldie ring and the order of every finite normal subgroup of G is a nonzero-divisor in R if and only if RG is a semiprime Goldie ring.*

PROOF. The proof is immediate using the construction in the proof of Theorem 2.6.

A right ideal of a ring R is said to be *essential* if it has nonzero intersection with every nonzero right ideal of R. A right ideal D of R is *dense* if for every $0\neq r_1 \in R$ and for every $r_2 \in R$ there exists $r \in R$ such that $r_1 r\neq 0$ and $r_2 r \in D$. *We denote the Jacobson radical of R by* Rad R. A right ideal A is said to be *small* if for every right ideal B, $A+B=R$ implies $B=R$. It is known that A is small if and only if $A\subset \text{Rad } R$.

The following remarks are well known.

REMARK 3.7. *A right ideal D is dense in R if and only if DG is dense in RG.*

REMARK 3.8. *A right ideal L is essential in R if and only if LG is essential in RG.*

A right ideal B is *rationally closed* in R if $x^{-1}B=\{r \in R \mid xr \in B\}$ is not dense for all $x \in R-B$. Let $I(R)$ denote the injective hull of R, then B is rationally closed in R if there exists a subset S of $I(R)$ such that $B=\{x \in R \mid Sx=0\}$ [**8**].

LEMMA 3.9. *A right ideal K of R is rationally closed in R if and only if KG is rationally closed in RG.*

PROOF. If K is rationally closed then there exists a subset $S\subset I(R)$ such that $K=\{x\in R|Sx=0\}$. We will show that $KG=\{x\in RG|SGx=0\}$. Let $x\in KG$ then $SGx=0$ since $Sk=0$ for all $k\in K$. Hence $x\in\{x\in RG|SGx=0\}$. Now suppose $0\neq x\notin KG$. We want to show there exists $y\in SG$ such that $yx\neq 0$. Let $x=r_1(g_1)g_1+\cdots+r_1(g_n)g_n$, since $x\notin KG$ there exists $r_i(g_i)$ such that $r_i(g_i)\notin K$. K is rationally closed so there exists $0\neq s\in S$ such that $sr_i(g_i)\neq 0$. Hence, $sx\neq 0$ implies $x\notin\{x\in RG|SGx=0\}$.

Conversely, suppose K is not rationally closed in R, then there exists $x\in R-K$ such that $x^{-1}K$ is dense in R. Thus $(x^{-1}K)G=x^{-1}KG$ is dense in RG and hence KG is not rationally closed in RG.

PROPOSITION 3.10 (RENAULT, [6]). *The group ring RG is self-injective if and only if R is self-injective and G is finite.*

PROOF. See [6].

LEMMA 3.11 (SHOCK, [9]). *Let R be a self-injective ring. Then R is a cogenerator if and only if R is right finite dimensional and Z(R) is rationally closed.*

PROOF. See Proposition 2 of [9].

If R is a self-injective ring then $Z(R)=\text{Rad } R$ [4]. It is known that if R is self-injective and finite dimensional then $R/\text{Rad } R$ is completely reducible.

THEOREM 3.12. *Let G be a finite group where the order of G is a unit in R, then R is a self-injective cogenerator ring if and only if RG is a self-injective cogenerator ring.*

PROOF. Let R be a self-injective cogenerator ring. It is clear that RG is finite dimensional and injective. By Lemma 3.11, we need only show that $Z(RG)$ is rationally closed. It is clear that if R contains no proper dense right ideals then every right ideal is rationally closed and conversely. So, we shall show that RG contains no proper dense right ideals. Let D be a dense right ideal of RG. Then $D+Z(R)G$ is dense and by Proposition 5.1 of [8], $(D+Z(R)G)/Z(R)G$ is dense in $RG/Z(R)G$ since $Z(R)G$ is rationally closed. Clearly, $RG/Z(RG)$ and $R/Z(R)$ are completely reducible. Therefore, $(R/Z(R))G\cong RG/Z(R)G$ is completely reducible [2] and thus $RG/Z(R)G$ contains no proper dense right ideals. Hence, $D+Z(R)G=RG$. But $Z(R)G\subset Z(RG)=\text{Rad } RG$ implies $Z(R)G$ is small. Hence, $D=RG$.

Conversely, let D be dense in R, $D\neq R$, then DG is dense in RG and $DG\neq RG$.

LEMMA 3.13 (SHOCK, [**9**]). *Suppose that $Z(Q(R))$ is the Jacobson radical of $Q(R)$ and is rationally closed. If $Q(R)/Z(Q(R))$ is a completely reducible ring and $R/Z(R)$ is semiprime, then R is a right order in $Q(R)$.*

PROOF. See Proposition 4 of [**9**].

THEOREM 3.14. *Let G be a finite group, then R is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of G is a zero-divisor in R if and only if RG is a right order in a self-injective cogenerator ring.*

PROOF. Let R be a right order in a self-injective cogenerator ring Q, then $Q=Q(R)$. By 3.6 of [**1**], we have $Q(RG)\simeq Q(R)G$ and thus by Theorem 3.12 $Q(RG)$ is a self-injective cogenerator ring. It is now clear that both $Q(RG)/Z(Q(RG))$ and $Q(R)/Z(Q(R))$ are completely reducible. Also, it is clear that $Q(R)G/Z(Q(R))G$ is completely reducible and that $RG/Z(R)G$ is semiprime. By Lemma 3.13 we need only to show that $RG/Z(RG)$ is semiprime. To do this, we first show that $Z(R)G=Z(RG)$. It is sufficient to show that $Z(Q(RG))=Z(Q(R))G$ since $Z(RG)=Z(Q(RG))\cap RG=Z(Q(R)G)\cap RG=Z(Q(R))G\cap RG=Z(R)G$. Now $(Q(R)/(Z(Q(R))))G\simeq Q(R)G/Z(Q(R))G\simeq Q(RG)/Z(Q(R))G$. Recall $Z(Q(R))G\subseteq Z(Q(RG))=\text{Rad}\, Q(RG)$. Hence, $Z(Q(R))G=Z(Q(RG))$ since $Q(RG)/Z(Q(R))G$ is completely reducible. The converse follows similarly.

In [**12**] Smith showed that if G is a poly- (cyclic or finite) group and R is a right order in a right Artinian ring then RG is a right order in a right Artinian ring. We extend this result to a class of group rings, where G need not be poly- (cyclic or finite), using a method of Small [**11**].

THEOREM 3.15. *Let G be a free abelian group. If R is a right order in a right Artinian ring then RG is a right order in a right Artinian ring.*

PROOF. It is clear that rad $(RG)=(\text{rad } R)G$ when G is free abelian. We now use the same argument as in Theorem 3.6 of [**10**].

REFERENCES

1. W. D. Burgess, *Rings of quotients of group rings*, Canad. J. Math. **21** (1969), 865–875. MR **39** #5714.

2. I. G. Connell, *On the group ring*, Canad. J. Math. **15** (1963), 650–685. MR **27** #3666.

3. A. W. Goldie, *Semiprime rings with maximum condition*, Proc. London Math. Soc. (3) **10** (1960), 201–220. MR **22** #2627.

4. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

5. D. S. Passman, *Infinite group rings*, Dekker, New York, 1971.

6. G. Renault, *Sur les anneaux de groupes*, C. R. Acad. Sci. Paris Sér. A–B **273** (1971), A84–A87. MR **44** #5387.

7. P. Ribenboim, *Rings and modules*, Interscience Tracts in Pure and Appl. Math., no. 24, Interscience, New York, 1969. MR **39** #4204.

8. R. C. Shock, *Injectivity, annihilators, and orders*, J. Algebra **19** (1971), 96–103. MR **43** #4861.

9. ———, *Orders in self-injective cogenerator rings*, Proc. Amer. Math. Soc. **35** (1972), 393–398.

10. ———, *Polynomial rings over finite dimensional rings*, Pacific J. Math. **42** (1972), 251–258.

11. L. W. Small, *Orders in Artinian rings*. II, J. Algebra **9** (1968), 266–273. MR **37** #6315.

12. P. F. Smith, *Quotient rings of group rings*, J. London Math. Soc. **3** (1971), 645–660.

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901