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ON A THEOREM CONCERNING BAIRE FUNCTIONS

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ABSTRACT. Mazurkiewicz, Sierpiński, and Kempisty proved that a function in Baire class 1 is the uniform limit of a sequence of functions each of which is the difference of two upper semicontinuous functions. A generalization of this theorem is shown to be a consequence of order and linear properties alone.

Mazurkiewicz [2], Sierpiński [3], and Kempisty [1] proved that a function in Baire class 1 is the uniform limit of a sequence of functions each of which is the difference of two upper semicontinuous functions. It is shown here that this theorem is a consequence of order and linear properties alone.

Suppose L is a Riesz space (lattice ordered vector space) which is σ-complete (every countable set of positive elements has a greatest lower bound). A sequence $f_1, f_2, \cdots$ of points is said to *order converge* to the point f if there exists a sequence $u_1 \geqq u_2 \geqq u_3 \geqq \cdots$ and a sequence $l_1 \leqq l_2 \leqq l_3 \leqq \cdots$ of points such that $\bigvee l_p = f$, $\bigwedge u_p = f$, and $l_p \leqq f_p \leqq u_p$. If L is the space of all real valued functions on a set then order convergence is equivalent to pointwise convergence. If L is an L_p space or the set of all measurable functions on a measure space then order convergence is equivalent to boundedness and convergence almost everywhere.

A sequence $f_1, f_2, f_3, \cdots$ of points of L is said to converge relatively uniformly to f if there exists an element g of L (called the regulator) such that if $\varepsilon > 0$, there exists a number N_ε such that if n is a positive integer greater than N_ε, then $-\varepsilon g \leqq f - f_n \leqq \varepsilon g$. Relative uniform convergence implies order convergence. Generally it is stronger.

Suppose C is a Riesz subspace of L and U is the set of all elements which are the greatest lower bound of a countable subset of C. The Riesz space version of the theorem of Mazurkiewicz, Sierpiński, and Kempisty is that if f is the order limit of a sequence of points of C then it is the relative uniform limit of a sequence of points each of which is the difference of two points of U. This is not true in general, e.g. let L be the space of *all bounded* functions on [0, 1] and C be the space of all continuous functions

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on [0, 1]. In Example 1 near the end of this paper it is shown that there is a bounded function in Baire class 1 which is not the uniform limit of any sequence of functions each of which is the difference of two bounded upper semicontinuous functions. Thus additional assumptions are required, some of which are discussed here.

For an important class of Riesz spaces, the regular spaces, relative uniform convergence is equivalent to order convergence and the theorem of Mazurkiewicz, Sierpiński, and Kempisty is trivially true. For a definition of regular Riesz spaces and a discussion of some of their properties see [4]. It is sufficient to note here that the class of regular Riesz spaces includes the spaces l_p $(1 \leqq p < \infty)$ and the spaces L_p $(1 \leqq p < \infty)$ of pth power summable functions with respect to a countably additive measure.

However if L is the space of all real valued functions on an infinite set X, then L is not regular, order convergence is weaker than relative uniform convergence, and the theorem of Mazurkiewicz, Sierpiński, and Kempisty is, in this instance, not immediate. This case will be covered in the theorem which follows.

Denote by θ the zero element of L. The point $e \geqq \theta$ of L is said to be a *weak unit* if for each point f of L, $f \wedge e = \theta$ only if $f = \theta$. The point $e \geqq \theta$ of L is said to have *property* c if whenever $h_1 \leqq h_2 \leqq h_3 \leqq \cdots$ is a sequence of elements of L such that $e = \bigvee h_i$ then there exists a point b of L such that $b \leqq \sum_{i=1}^{n} h_i$ for every positive integer n.

It is shown in this paper that if e has property c then it is a weak unit. It can be shown by use of a representation theorem that if e is a weak unit of L then L can be embedded in a σ-complete Riesz space V such that e has property c in V.

THEOREM 1. *Suppose L is a σ-complete Riesz space, e is a point of L with property* c, *C is a Riesz subspace of L containing e, and U is the set of all points which are the greatest lower bounds of a countable subset of C. Then if f is the order limit of a sequence of points of C, it is the relative uniform limit, with regulator e, of a sequence of points each of which is the difference of two points of U.*

PROOF. It will be sufficient to show that if k is a positive number there is a point of $U-U$ in the order interval $[f, f+ke]$.

Let $f+ke=g$. There is a sequence $\{f_p\}$ of points of C order converging to f. Let $y_i = \bigvee_{p \geqq i} f_p$ and $z_i = \bigwedge_{p \geqq i} f_p + ke$. Then $-y_i \in U$ and $z_i \in U$. Also $f = \bigwedge y_i$ and $g = \bigvee z_i$.

Let $\alpha_1 = -y_1$ and for each positive integer i let

$$\beta_i = (z_i - y_i) \wedge \theta \quad \text{and} \quad \alpha_{i+1} = (z_i - y_{i+1}) \wedge \theta.$$

Each of α_i and β_i is in U. Now $ke = \bigvee (z_i - y_i)$ and $ke = \bigvee (z_i - y_{i+1})$. As e

has property c, $\{\sum_{j=1}^{n} \alpha_j\}$ and $\{\sum_{j=1}^{n} \beta_j\}$ each have a lower bound in L. By hypothesis there exists an $a \leqq \theta$ and a $b \leqq \theta$ both in U such that

$$b = \bigwedge \sum_{j=1}^{n} \beta_j \quad \text{and} \quad a = \bigwedge \sum_{j=1}^{n} \alpha_j.$$

Now to show $g \geqq b-a \geqq f$.

Let $m_i = \sum_{j=1}^{i+1} -\alpha_j$ and $n_i = \sum_{j=1}^{i} \beta_j$.

To show that $b-a \geqq f$, the following facts will be derived:

(a) $n_{i+1}+m_{i+1} \leqq n_i+m_i$,

(b) $\bigwedge (n_i+m_i) = b-a$, and

(c) $n_i+m_i \geqq y_{i+1}$.

As $f = \bigwedge y_i$, statements (b) and (c) are all that is necessary. Statement (a) is used in the argument for statement (b).

The following identity, which is valid in all vector lattices, will be used repeatedly: $u-(v \vee u) = (u-v) \wedge \theta$.

(a) If i is a positive integer

$$\begin{aligned} n_{i+1} + m_{i+1} - (n_i + m_i) &= (z_{i+1} - y_{i+1}) \wedge \theta - (z_{i+1} - y_{i+2}) \wedge \theta \\ &= z_{i+1} - (z_{i+1} \vee y_{i+1}) - z_{i+1} + (z_{i+1} \vee y_{i+2}) \\ &= (z_{i+1} \vee y_{i+2}) - (z_{i+1} \vee y_{i+1}). \end{aligned}$$

Now $y_{i+2} \leqq y_{i+1}$, so $z_{i+1} \vee y_{i+2} \leqq z_{i+1} \vee y_{i+1}$. Thus $\theta \geqq n_{i+1}+m_{i+1}-(n_i+m_i)$ and $n_i+m_i \geqq n_{i+1}+m_{i+1}$.

(b) If i is a positive integer,

$$b + m_i \leqq b + m_{i+1} \leqq n_{i+1} + m_{i+1} \leqq n_i + m_i.$$

Thus n_i+m_i is an upper bound for $b+m_p$ and $n_i+m_i \geqq \bigvee (b+m_p) = b-a$. Then as $b-a \leqq n_i+m_i \leqq n_i-a$ and $b-a = \bigwedge n_i-a$, it follows that $\bigwedge (n_i+m_i) = b-a$.

(c) If i is a positive integer,

$$\begin{aligned} n_i + m_i &= y_1 + \sum_{j=1}^{i} \{[(z_j - y_j) \wedge \theta] - [(z_j - y_{j+1}) \wedge \theta]\} \\ &= y_1 + \sum_{j=1}^{i} \{[-y_j - (-y_j \vee -z_j)] - [-y_{j+1} - (-y_{j+1} \vee -z_j)]\} \\ &= y_1 + \sum_{j=1}^{i} (y_{j+1} - y_j) + \sum_{j=1}^{i} [(-y_{j+1} \vee -z_j) - (-y_j \vee -z_j)] \\ &= y_{i+1} + \sum_{j=1}^{i} [(y_j \wedge z_j) - (y_{j+1} \wedge z_j)]. \end{aligned}$$

As $y_j \geqq y_{j+1}$, $y_j \wedge z_j \geqq y_{j+1} \wedge z_j$, and $[(y_j \wedge z_j) - (y_{j+1} \wedge z_j)] \geqq \theta$. So

$$n_i + m_i = y_{i+1} + \sum_{j=1}^{i} [(y_j \wedge z_j) - (y_{j+1} \wedge z_j)] \geqq y_{i+1} + \sum_{j=1}^{i} \theta = y_{i+1}.$$

This completes the proof that $b - a \geqq f$. An analogous argument shows that $b - a \leqq g$.

In the following two theorems the Riesz space L is not necessarily assumed to be σ-complete but is Archimedean. The statement that L is *Archimedean* means that for each point $f \geqq \theta$ of L, $\bigwedge (1/n)f = \theta$. The property of σ-completeness implies the property of being Archimedean.

The proof for Theorem 2 was suggested by the referee. It is simpler then the author's original argument.

THEOREM 2. *Suppose L is an Archimedean Riesz space and e is a point of L with property* c. *Then e is a weak unit.*

PROOF. Suppose $x \in L$ and $x \wedge e = \theta$. Define $x_n = e - (1/n)x$ for each positive integer n. Then $x_1 \leqq x_2 \leqq x_3 \cdots$ and $\bigvee x_n = e$. Now for each positive integer k, $\sum_{n=1}^{k} x_n = ke - \sum_{n=1}^{k} (1/n)x$. As e has property c, there exists a point b of L such that $b \leqq ke - \sum_{n=1}^{k} (1/n)x$. Since $x \wedge e = \theta$, $b \leqq ke$ and $b \leqq -\sum_{n=1}^{k} (1/n)x$. Thus, $-b \geqq (\sum_{n=1}^{k} 1/n)x$. The series $\sum 1/n$ increases without bound, so that for every positive integer p it is true that $-b \geqq px$. Since L is Archimedean, this implies that $x = \theta$.

The statement that e is a *strong unit* of L means that $e \geqq \theta$ and if f is in L there is a positive number k such that $ke \geqq f$. If e is a strong unit, then it is a weak unit.

In the following, by use of a representation theorem for Archimedean vector lattices, L will be assumed to be a subspace of the set of all continuous extended real valued functions on an extremally disconnected compact Hausdorff space S which are finite except possibly on a nowhere dense subset of S.

If L is assumed to have a weak unit e, then e can be taken to be the function identically equal to 1 on S. Further, if L is assumed to have a strong unit then L can be assumed to consist of only finite valued functions on S.

THEOREM 3. *Suppose L is an Archimedean Riesz space, e is a point of L with property* c *and g is a strong unit of L. Then e is a strong unit and g has property* c. *Further, if L is σ-complete, it is finite dimensional.*

PROOF. As $g \geqq \theta$, $\{e - (1/p)g\}$ is a nondecreasing sequence such that $\bigvee (e - (1/p)g) = e$. Thus there is an element $k \in L$ such that $k \leqq \sum_{p=1}^{n} e - (1/p)g$ for every positive integer n. There is a negative integer r such that

$rg \leqq k$. So $\sum_{p=1}^{n} e-(1/p)g = ne - g \sum_{p=1}^{n} 1/p \geqq rg$. Or, $ne \geqq g(r+\sum_{p=1}^{n}(1/p))$. Now as mentioned above g may be taken to be the function which is 1 at every point of S. Also, let n be an integer such that $\sum_{p=1}^{n}(1/p) \geqq -r+1$. Thus $ne \geqq (r+\sum_{p=1}^{n}(1/p)) \geqq 1 = g$. Therefore e is a strong unit.

Let $g_1 \leqq g_2 \leqq g_3 \leqq \cdots$ be a sequence of elements of L such that $g = \bigvee g_i$. There exists a positive integer k such that $kg \geqq e$. Then $\{[k(g_i \vee \theta)] \wedge e + g_i \wedge \theta\}$ is a nondecreasing sequence with supremum e. Thus there is an element $a \in L$ such that $\sum_{i=1}^{\infty} g_i \wedge \theta \geqq a$.

Suppose L is σ-complete and e has property c, is a strong unit, and is equal to 1 at each point of S.

Let Ω be the collection of all subsets ω of S such that the characteristic function of ω is in L. Each set in Ω is open and closed and if ω is in Ω then ω' is in Ω. Further the union and intersection of any two sets in Ω is in Ω. Suppose $\omega_1, \omega_2, \omega_3, \cdots$ is a countable increasing tower of sets in Ω. The characteristic function of $\mathrm{cl}(\bigcup \omega_i)$ is the supremum of the characteristic functions of $\omega_1, \omega_2, \cdots$ and must belong to L as L is σ-complete. Thus $\mathrm{cl}(\bigcup \omega_i)$ is in Ω as is $\omega_0 = (\mathrm{cl}(\bigcup \omega_i))'$. Then $\omega_1 \cup \omega_0, \omega_2 \cup \omega_0, \omega_3 \cup \omega_0, \cdots$ is an increasing tower of sets of Ω whose union is dense in S.

Suppose x_0 is a point of S which is not in $\bigcup(\omega_i \cup \omega_0)$. For each positive integer i let f_i be the function which is 1 at each point of $\omega_i \cup \omega_0$ and -1 otherwise. Then $\bigvee f_i = e$ and each f_i is in L. But as $\sum_{i=1}^{n} f_i(x_0) \to -\infty$ as $n \to \infty$ and each function in L is bounded, there is no a in L such that $a \leqq \sum_{i=1}^{n} f_i$ for each positive integer n. This contradicts the fact that e has property c. Therefore, $\bigcup(\omega_i \cup \omega_0) = S$. As S is compact, some finite subtower covers S. This implies that the original tower is finite and, therefore, that Ω is finite, for if Ω were infinite a countably infinite tower of sets in Ω could be formed.

Suppose h is in L. Then $\bigvee \{[n(h \vee \theta)] \wedge e\}$, $n = 1, 2, 3, \cdots$, is in L and is the characteristic function of the closure of $\{x \mid h(x) > 0\}$.

Let Ω' be the subcollection of Ω consisting of all the sets of Ω which contain no other set in Ω. Then Ω' is a partition of S. Suppose k is in L and k has the values y_1 and y_2 $(y_1 < y_2)$ on some set ω of Ω'. Let $h = k - (y_1 + y_2)/2$ and γ be the closure of $\{x \mid h(x) > 0\}$. Thus γ contains $k^{-1}(y_2)$ but not $k^{-1}(y_1)$ and as pointed out above γ is in Ω'. But then $\gamma \cap \omega$ is in Ω' and is a proper subset of ω. This is a contradiction. Therefore k is constant on ω and it follows that k is a linear combination of the characteristic functions of the sets in Ω'. Thus L is finite dimensional.

Example 1. In the following all complements are taken with respect to $[0, 1]$. Let M_1 be a Cantor set in $[0, 1]$. For each positive integer $i > 1$ take a Cantor set in each component of the complement of $M_1 \cup \cdots \cup M_{i-1}$ and let M_i be the union of these Cantor sets. Then for each positive integer i let $H_i = M_i \cup [1/i, 1]$. Suppose x is a number in $[0, 1]$ and p is the smallest positive integer such that x is in H_p. Let $f(x)=-p$ and $g(x)=-(p+1)$ if p is odd and $g(x)=-p$ if p is even.

Each of f and g is upper semicontinuous. The function $f-g$ takes on the values 0 and 1. If each of f_1 and g_1 is upper semicontinuous and $|f(x)-g(x)-(f_1(x)-g_1(x))|<\frac{1}{4}$ for every number x in [0, 1], then each of f_1 and g_1 is unbounded below.

An indication of the proof of the last statement is the following: Suppose p is an odd positive integer, k is a number such that $0<k<1/(p+2)$, and $w \in M_p \cap (0, k)$. Since $f_1(x)-g_1(x)>\frac{3}{4}$ for $x \in M_p \cap (0, k)$, $g_1(w)<f_1(w)-\frac{3}{4}$. As g_1 is upper semicontinuous there is an open interval $(a, b) \subset (0, k)$ containing w such that every point of g_1 with abscissa in (a, b) is beneath the horizontal line with ordinate $f_1(w)-\frac{3}{4}$. Because w is a limit point of M_{p+1}, (a, b) contains a number v of M_{p+1}. Thus $g_1(v)<f_1(w)-\frac{3}{4}$. Therefore since $f_1(v)-g_1(v)<\frac{1}{4}$, $f_1(v)<f_1(w)-\frac{1}{2}$. As f_1 is upper semicontinuous, there is an open interval $(c, d) \subset (a, b)$ containing v such that every point of f_1 with abscissa in (c, d) is beneath the horizontal line with ordinate $f_1(w)-\frac{1}{2}$. Because v is a limit point of M_{p+2}, (c, d) contains a number u of M_{p+2} and $f_1(u)<f_1(w)-\frac{1}{2}$.

The following example was suggested by the referee.

EXAMPLE 2. Let X be an uncountable set and let L be the space of all real valued functions on X that are constant on the complement of some finite set depending on the function. Then L is not σ-complete, it is infinite dimensional and the constant function 1 is a strong unit and has property c.

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