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ON $\sum_{n\leqq x}\sigma^*(n)$ AND $\sum_{n\leqq x}\varphi^*(n)$

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ABSTRACT. Let $\sigma^*(n)$ and $\varphi^*(n)$ be the unitary analogues of $\sigma(n)$ and $\varphi(n)$ respectively. It is known that $E(x)=\sum_{n\leqq x}\sigma^*(n)-(\pi^2x^2/12\zeta(3))=O(x\log^2 x)$ and

$$F(x)=\sum_{n\leqq x}\varphi^*(n)-\tfrac{1}{2}\alpha x^2=O(x\log^2 x),$$

where α is a positive constant. In this paper we improve the order estimates of $E(x)$ and $F(x)$ to $E(x)=O(x\log^{5/3}x)$ and

$$F(x)=O(x\log^{5/3}x(\log\log x)^{4/3}).$$

1. **Introduction.** It is well known that a divisor $d>0$ of the positive integer n is called unitary, if $d\delta=n$ and $(d,\delta)=1$. Let $\sigma^*(n)$ denote the sum of the unitary divisors of n. For integers $a, b, b>0$, let $(a,b)^*$ denote the greatest divisor of a which is a unitary divisor of b. If $(a,b)^*=1$, then a is said to be semiprime to b. The symbol $(a,b)^*$ was introduced by E. Cohen (cf. [**1**, §1]). Let $\varphi^*(n)$ denote the number of positive integers $a\leqq n$ such that a is semiprime to n.

E. Cohen (cf. [**1**, Corollaries 4.1.1 and 4.1.2]) established the following asymptotic formulae:

$$\sum_{n\leqq x}\sigma^*(n)=\frac{\pi^2x^2}{12\zeta(3)}+O(x\log^2 x), \tag{1.1}$$

where $\zeta(s)$ denotes the Riemann Zeta function defined by $\zeta(s)=\sum_{n=1}^{\infty}1/n^s$ for $s>1$;

$$\sum_{n\leqq x}\varphi^*(n)=\tfrac{1}{2}\alpha x^2+O(x\log^2 x), \tag{1.2}$$

where

$$\alpha=\prod_p\left(1-\frac{1}{p(p+1)}\right), \tag{1.3}$$

the product being extended over all primes p.

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The object of the present paper is to improve the order estimates of the error terms in the asymptotic formulae (1.1) and(1.2). In fact, we prove that

$$\sum_{n\leq x}\sigma^*(n) = \frac{\pi^2x^2}{12\zeta(3)} + O(x\log^{5/3}x), \tag{1.4}$$

$$\sum_{n\leq x}\varphi^*(n) = \tfrac{1}{2}\alpha x^2 + O(x\log^{5/3}x(\log\log x)^{4/3}), \tag{1.5}$$

where α is the constant given by (1.3).

2. **Prerequisites.** In this section, we prove some lemmas which are needed in the present discussion. Let $\mu(n)$ denote the Möbius function, $\varphi(n)$ denote the Euler totient function, $\sigma(n)$ denote the sum of the divisors of n and $\gamma(n)$ denote the core (the maximal square-free divisor) of n. Let $a(n)$ be the arithmetical function defined by $a(1)=1$ and $a(n)=\prod_{i=1}^r(\alpha_i-1)$ if $n=\prod_{i=1}^r p_i^{\alpha_i}$.

It is known (cf. [**1**, Corollary 2.2.1]) that $\varphi^*(n)$ is multiplicative, and so we have

$$\varphi^*(n) = n\prod_{i=1}^r\left(1-\frac{1}{p_i^{\alpha_i}}\right), \tag{2.1}$$

if $n=\prod_{i=1}^r p_i^{\alpha_i}$.

We use the following best-known estimates due to A. Walfisz [**4**] concerning the averages of the functions $\sigma(n)$ and $\varphi(n)$:

Lemma 2.1 (cf. [**4**, Satz 4, p. 99]). *For* $x\geqq 2$,

$$\sum_{n\leq x}\sigma(n) = \frac{\pi^2x^2}{12} + O(x\log^{2/3}x). \tag{2.2}$$

Lemma 2.2 (cf. [**4**, Satz 1, p. 144]). *For* $x\geqq 3$,

$$\sum_{n\leq x}\varphi(n) = \frac{3x^2}{\pi^2} + O(x\log^{2/3}x(\log\log x)^{4/3}). \tag{2.3}$$

Lemma 2.3. $\sigma^*(n)=\sum_{d^2\delta=n}\mu(d)\,d\sigma(\delta)$.

Proof. Since $\mu(n)n$ and $\sigma(n)$ are multiplicative, it follows (cf. [**3**, Lemma 2.4]) that $\sum_{d^2\delta=n}\mu(d)\,d\sigma(\delta)$ is a multiplicative function of n. Also, $\sigma^*(n)$ is multiplicative (cf. [**1**, Lemma 6.1]). Hence, it is enough if we verify the identity for $n=p^\alpha$, a prime power. We have

$$\sum_{d^2\delta=p}\mu(d)\,d\sigma(\delta) = \mu(1)\sigma(p) = 1+p = \sigma^*(p),$$

and for $\alpha\geqq 2$,

$$\begin{aligned}\sum_{^2\delta=p^\alpha} \mu(d)\, d\sigma(\delta) &= \mu(1)\sigma(p^\alpha) + \mu(p)p\sigma(p^{\alpha-2}) \\ &= (1 + p + \cdots + p^\alpha) - p(1 + p + \cdots + p^{\alpha-2}) \\ &= 1 + p^\alpha = \sigma^*(p^\alpha).\end{aligned}$$

Hence the lemma follows.

LEMMA 2.4. $\varphi^*(n)=\sum_{d\delta=n} (\varphi(d)a(\delta)\varphi(\delta)\gamma(\delta)/\delta)$.

PROOF. Since $a(n)$, $\gamma(n)$, $\varphi(n)$ and $\varphi^*(n)$ are multiplicative, it is enough, if we verify the identity for $n=p^\alpha$, a prime power. We have

$$\begin{aligned}&\sum_{d\delta=p^\alpha} \frac{\varphi(d)a(\delta)\varphi(\delta)\gamma(\delta)}{\delta} \\ &\quad= \varphi(1)a(p^\alpha)(p-1) + \varphi(p)a(p^{\alpha-1})(p-1) + \cdots \\ &\qquad + \varphi(p^{\alpha-2})a(p^2)(p-1) + \varphi(p^\alpha)a(1) \\ &\quad= (p-1)\{(\alpha-1) + p(1-1/p)(\alpha-2) + \cdots \\ &\qquad\qquad + p^{\alpha-2}(1-1/p)\} + \varphi(p^\alpha) \\ &\quad= (p-1)(\alpha-1) + (p-1)^2 \\ &\qquad \times \{(\alpha-2) + p(\alpha-3) + \cdots + p^{\alpha-3}\} + p^\alpha(1-1/p) \\ &\quad= (p-1)(\alpha-1) + (p-1)^2 \\ &\qquad \times \left\{\frac{\alpha(p^{\alpha-2}-1)}{p-1} + \frac{1}{p} - \frac{1-\alpha p^{\alpha-1} + (\alpha-1)p^\alpha}{p(p-1)^2}\right\} + (p^\alpha - p^{\alpha-1}) \\ &\quad= (p-1)(\alpha-1) + (p^{\alpha-1} + p + \alpha - \alpha p - 2) + (p^\alpha - p^{\alpha-1}) \\ &\quad= p^\alpha - 1 = \varphi^*(p^\alpha).\end{aligned}$$

Hence the lemma follows.

LEMMA 2.5. $A(x)=\sum_{n\leqq x} a(n)\gamma(n)=O(x)$.

PROOF. We have $1/\zeta(s)=\prod_p (1-1/p^s)$ for $s>1$ (cf. [**2**, Theorem 280]). Hence

$$\begin{aligned}&\frac{1}{\zeta(2s-1)}\sum_{n=1}^{\infty} \frac{a(n)\gamma(n)}{n^s} \\ &\qquad= \prod_p \left\{\left(1-\frac{p}{p^{2s}}\right)\left(1+\frac{p}{p^{2s}}+\frac{2p}{p^{3s}}+\frac{3p}{p^{4s}}+\cdots\right)\right\} \\ &\qquad= \prod_p \left\{1+\frac{2p}{p^{3s}}+\frac{3p-p^2}{p^{4s}}+\frac{4p-2p^2}{p^{5s}}+\cdots\right\} \\ &\qquad= \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \qquad \text{say.}\end{aligned}\tag{2.4}$$

Hence

$$\sum_{n=1}^{\infty} \frac{a(n)\gamma(n)}{n^s} = \zeta(2s-1)\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \sum_{n=1}^{\infty}\frac{n}{n^{2s}}\sum_{n=1}^{\infty}\frac{b(n)}{n^s},$$

so that

$$a(n)\gamma(n) = \sum_{d^2\delta=n} db(\delta). \tag{2.5}$$

Also, by (2.4) we see that the abscissa of absolute convergence β of $\sum_{n=1}^{\infty}(b(n)/n^s)$ is $\beta=\frac{3}{4}$. Hence, we have

$$\sum_{n\leqq x} b(n) = O(x^{3/4+\varepsilon}), \qquad \text{for every } \varepsilon>0. \tag{2.6}$$

Hence by (2.5) and (2.6), we have

$$\begin{aligned}\sum_{n\leqq x} a(n)\gamma(n) &= \sum_{d^2\delta\leqq x} db(\delta) = \sum_{d\leqq x^{1/2}} d \sum_{\delta\leqq x/d^2} b(\delta)\\ &= O\left(\sum_{d\leqq x^{1/2}} d\left(\frac{x}{d^2}\right)^{3/4+\varepsilon}\right)\\ &= O\left(x^{3/4+\varepsilon}\sum_{d\leqq x^{1/2}}\frac{1}{d^{1/2+2\varepsilon}}\right)\\ &= O(x^{3/4+\varepsilon}(x^{1/2})^{1/2-2\varepsilon}) = O(x).\end{aligned}$$

LEMMA 2.6. $\sum_{n=1}^{\infty}(a(n)\varphi(n)\gamma(n)/n^3)=\zeta(2)\alpha$, where α is given by (1.3).

PROOF. Applying Lemma 2.5 and partial summation, it is easy to show that $\sum_{n\leqq x}(a(n)\gamma(n)/n^2)=O(1)$. Now since $\varphi(n)\leqq n$, it follows that the series $\sum_{n=1}^{\infty}(a(n)\varphi(n)\gamma(n)/n^3)$ is absolutely convergent. Further, the general term of the series is multiplicative. Hence the series can be expanded into an infinite product of Euler type (cf. [2, Theorem 286]), so that we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a(n)\varphi(n)\gamma(n)}{n^s} &= \prod_p\left\{1+\sum_{i=2}^{\infty}\frac{(i-1)(p-1)}{p^{2i}}\right\}\\ &= \prod_p\left\{1+\frac{p-1}{p^4(1-p^{-2})^2}\right\} = \frac{\prod_p(1-2/p^2+1/p^3)}{\prod_p(1-1/p^2)^2}\\ &= \zeta^2(2)\prod_p\left(1-\frac{2}{p^2}+\frac{1}{p^3}\right) = \zeta(2)\prod_p\left(1-\frac{1}{p(p+1)}\right).\end{aligned}$$

Hence the lemma follows.

LEMMA 2.7. *For* $x\geqq 2$,

$$\sum_{n>x}\frac{a(n)\gamma(n)}{n^2} = O\left(\frac{1}{x}\right) \tag{2.7}$$

and

$$\sum_{n\leq x}\frac{a(n)\gamma(n)}{n}=O(\log x). \tag{2.8}$$

PROOF. Both (2.7) and (2.8) follow by partial summation and Lemma 2.5.

3. Proofs of (1.4) and (1.5). By Lemmas 2.3 and 2.1, we have

$$\begin{aligned}
\sum_{n\leq x}\sigma^*(n) &= \sum_{d^2\delta\leq x}\mu(d)\,d\sigma(\delta)=\sum_{d\leq x^{1/2}}\mu(d)d\sum_{\delta\leq x/d^2}\sigma(\delta)\\
&=\sum_{d\leq x^{1/2}}\mu(d)\,d\left\{\frac{\pi^2}{12}\left(\frac{x}{d^2}\right)^2+O\left(\frac{x}{d^2}\log^{2/3}\left(\frac{x}{d^2}\right)\right)\right\}\\
&=\frac{\pi^2x^2}{12}\sum_{n\leq x^{1/2}}\frac{\mu(n)}{n^3}+O\left(x\log^{2/3}x\sum_{n\leq x^{1/2}}\frac{1}{n}\right)\\
&=\frac{\pi^2x^2}{12}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^3}+O\left(x^2\sum_{n>x^{1/2}}\frac{1}{n^3}\right)+O(x\log^{5/3}x)\\
&=\frac{\pi^2x^2}{12\zeta(3)}+O(x)+O(x\log^{5/3}x),
\end{aligned}$$

since $\sum_{n=1}^{\infty}(\mu(n)/n^3)=1/\zeta(3)$ (cf. [**2**, Theorem 287]). Hence (1.4) follows.

By Lemmas 2.4 and 2.2, we have

$$\begin{aligned}
\sum_{n\leq x}\varphi^*(n) &= \sum_{d\delta\leq x}\frac{\varphi(d)a(\delta)\varphi(\delta)\gamma(\delta)}{\delta}\\
&=\sum_{\delta\leq x}\frac{a(\delta)\varphi(\delta)\gamma(\delta)}{\delta}\sum_{d\leq x/\delta}\varphi(d)\\
&=\sum_{\delta\leq x}\frac{a(\delta)\varphi(\delta)\gamma(\delta)}{\delta}\left\{\frac{3x^2}{\pi^2\delta^2}+O\left(\frac{x}{\delta}\log^{2/3}\left(\frac{x}{\delta}\right)\left(\log\log\left(\frac{x}{\delta}\right)\right)^{4/3}\right)\right\}\\
&=\frac{3x^2}{\pi^2}\sum_{n\leq x}\frac{a(n)\varphi(n)\gamma(n)}{n^3}\\
&\quad+O\left(x\log^{2/3}x(\log\log x)^{4/3}\sum_{n\leq x}\frac{a(n)\varphi(n)\gamma(n)}{n^2}\right)\\
&=\frac{x^2}{2\zeta(2)}\sum_{n=1}^{\infty}\frac{a(n)\varphi(n)\gamma(n)}{n^3}+O\left(x^2\sum_{n>x}\frac{a(n)\gamma(n)}{n^2}\right)\\
&\quad+O\left(x\log^{2/3}x(\log\log x)^{4/3}\sum_{n\leq x}\frac{a(n)\gamma(n)}{n}\right),
\end{aligned} \tag{3.1}$$

since $\varphi(n)\leq n$. Now, the first O-term in (3.1) is $O(x^2\cdot 1/x)=O(x)$, by

(2.7) and the second O-term in (3.1) is $O(x \log^{5/3} x(\log \log x)^{4/3})$, by (2.8). Hence (1.5) follows by Lemma 2.6.

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