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BERNOULLI SHIFTS ARE DETERMINED BY THEIR FACTOR ALGEBRAS[1]

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We shall show that, if S is an invertible measure-preserving transformation of the unit interval with the same factor algebras as a Bernoulli shift T, then S is isomorphic to T. This answers affirmatively a conjecture by G.-C. Rota in 1968.

DEFINITIONS. A *transformation* T is an invertible measure-preserving transformation of the unit interval X with Lebesgue sets Σ and Lebesgue measure μ. T is called *ergodic* if there are no T-invariant sets $A \in \Sigma$ with $0<\mu(A)<1$. A *factor algebra* of T is a complete σ-algebra $\Sigma_1 \subseteq \Sigma$ such that $T\Sigma_1=T^{-1}\Sigma_1=\Sigma_1$. A *partition* P is a countable disjoint collection of measureble sets whose union is X. The join of P and Q is $P \vee Q=\{p\cap q \mid p \in P, q \in Q\}$ and $\bigvee_k^n T^iP=T^kP\vee T^{k+1}P\vee\cdots\vee T^nP$. Also, $\bigvee_{-\infty}^{\infty} T^iP$ denotes the smallest factor algebra containing P. P is a generator for T if $\bigvee_{-\infty}^{\infty} T^iP=\Sigma$ and $\{T^iP\}$ is independent if, for each $n>0$, P is independent of $\bigvee_1^n T^iP$. T is a *Bernoulli shift* if it has a generator P such that $\{T^iP\}$ is independent.

The entropy $H(P)=-\sum_{p\in P} \mu(p)\log \mu(p)$, the relative entropy

$$H(T, P) = \lim n^{-1}H\left(\bigvee_1^n T^iP\right) \quad \text{and} \quad H(T) = \sup_P H(T, P).$$

We have

(1) $H(T, P)\leqq H(P)$ with equality iff $\{T^iP\}$ is independent.

(2) If P is a generator for T, then $H(T)=H(T, P)$.

(3) If T is ergodic and $\varepsilon>0$, there is a generator P such that $H(P)\leqq H(T)+\varepsilon$.

The reader is referred to [3] for details about entropy. The result (3) is due to Rohlin and is the primary tool in our proof of the following theorem.

THEOREM. *If S has the same factor algebras as a Bernoulli shift T, then S is isomorphic to T.*

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PROOF. We begin by making two elementary observations:

(4) S is ergodic.

(5) $\bigvee_{-\infty}^{\infty} T^iP=\bigvee_{-\infty}^{\infty} S^iP$ for any P.

To prove (4), note that, if $SA=A$, then $\{\Phi, A, X-A, X\}$ is a factor algebra of S, hence of T. Since T^2 is ergodic, this implies that $\mu(A)$ is 0 or 1. The result (5) follows from the fact that P is contained in the factor algebra $\bigvee_{-\infty}^{\infty} T^iP$ of T, hence so is $\bigvee_{-\infty}^{\infty} S^iP$.

We now prove

(6) $H(S)=H(T)$.

First use (3) to choose a generator P for S such that $H(P)\leqq H(S)+\varepsilon$. P must be a generator for T, from (5), and (1) and (2) give

$$H(T) \leqq H(P) \leqq H(S) + \varepsilon.$$

Now interchange the roles of S and T to obtain (6).

To complete the proof of the theorem, choose a generator P for T such that $\{T^iP\}$ is independent. We then have $H(T)=H(T, P)=H(P)$, and P is a generator for S. Therefore,

$$H(S) = H(S, P) \leqq H(P) = H(T) = H(S),$$

so (1) implies that $\{S^iP\}$ is independent.

REMARKS. 1. The result (6) only uses the fact that T^2 is ergodic and can be easily proved assuming only that T is ergodic.

2. Any two irrational rotations of the unit circle have the same factor algebras [**1**].

3. If T is a K-automorphism [**3**], then S must also be a K-automorphism. However, since there is a K-automorphism T which is not isomorphic to T^{-1} (see [**2**]), the theorem is false for K-automorphisms.

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