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RECAPTURING A HOLOMORPHIC FUNCTION ON AN ANNULUS FROM ITS MEAN BOUNDARY VALUES

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ABSTRACT. Let D be an annulus in the complex plane with closure $\bar{D}$ and boundary ∂D. We prove that a function f, holomorphic in D with $C^{1+\varepsilon}(\partial D)$ boundary data for some $\varepsilon>0$, is uniquely determined by its arithmetic means $s_n(f)$ and $s_{0n}(f)$ over equally spaced points on ∂D. We also give an explicit formula for recapturing f from its means $s_n(f)$ and $s_{0n}(f)$. Furthermore, we derive the relations between $s_n(f)$ and $s_{0n}(f)$ which are necessary and sufficient for the analytic continuability of f from D to the whole disc.

1. **Introduction.** Let $U:|z|<1$ be the open unit disc and $T:|z|=1$ be the unit circle in the complex plane. For an $\varepsilon>0$, we let $A^{1+\varepsilon}(U)$ denote the class of all functions

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

with $a_n=O(1/n^{1+\varepsilon})$. If f is a continuous function on T, we consider the arithmetic means

$$s_n(f)=\frac{1}{n}\sum_{k=1}^{n} f(w_n^k),$$

$n=1,2,\cdots$, of f on T, where $w_n^k=\exp(i2\pi k/n)$ are the nth roots of unity. It is known (cf. [**1**]) that if $f\in A^{1+\varepsilon}(U)$ then the sequence $\{s_n(f)\}$ uniquely determines f in $A^{1+\varepsilon}(U)$. Also, an explicit representation of a function f in $A^{1+\varepsilon}(U)$ in terms of the sequence $\{s_n(f)\}$ is given in [**3**]. In this paper, we establish these results for functions holomorphic in an annulus. Hence, one can explicitly recapture a function f, holomorphic in a simply connected or doubly connected domain G and continuous on the closure of G, from its "means" on the boundary ∂G of G, provided that an explicit conformal map of G onto the unit disc or an annulus can be found and has a sufficiently smooth extension to ∂G and that f is sufficiently smooth on ∂D.

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Let $0<r_0<1$, and consider the annulus $D=\{z:r_0<|z|<1\}$. For an $\varepsilon>0$, we denote by $A^{1+\varepsilon}(D)$ the class of all functions $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ such that for $n>0$, $a_n=O(1/n^{1+\varepsilon})$ and $a_{-n}=O(r_0^n/n^{1+\varepsilon})$. If f is a function continuous on the boundary ∂D of D, we define (cf. [2]) the *Riemann coefficients* of f by

$$R_n(f) = s_n(f) - s_\infty(f) \quad \text{and} \quad R_{0n}(f) = s_{0n}(f) - s_{0\infty}(f),$$

where

$$s_{0n}(f) = \frac{1}{n}\sum_{k=1}^{n} f(r_0 w_n^k), \qquad n = 1, 2, \cdots,$$

and

$$s_\infty(f) = \lim_{n\to\infty} s_n(f), \qquad s_{0\infty}(f) = \lim_{n\to\infty} s_{0n}(f).$$

For all functions f "smooth" on ∂D, it is known (cf. [2]) that the Riemann coefficients $R_n(f)$ and $R_{0n}(f)$ have similar asymptotic decay as the Fourier coefficients $a_n(f)$ and $a_{0n}(f)$ respectively, where

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt \quad \text{and} \quad a_{0n}(f) = \frac{1}{2\pi}\int_0^{2\pi} f(r_0e^{it})e^{-int}\,dt.$$

It is also known (cf. [8, p. 6]) that f is holomorphic in D if and only if $a_{0n}(f)=a_n(f)r_0^n$ for all $n=0, \pm 1, \cdots$. On the other hand, it is easy to see that for functions f holomorphic in D, $R_n(f)$ and $R_{0n}(f)$ are not related, since there are rational functions q_n and q_{0n} satisfying $R_m(q_n)=\delta_{m,n}$, $R_{0n}(q_m)=0$, $R_m(q_{0n})=0$ and $R_{0m}(q_{0n})=\delta_{m,n}$ for all m and n. However, we will give the relations between $R_n(f)$ and $R_{0n}(f)$ which are necessary and sufficient for functions $f\in A^{1+\varepsilon}(D)$ to be of class $A^{1+\varepsilon}(U)$.

2. **Uniqueness, representation and analytical continuability theorems.** We first establish the following uniqueness theorem.

THEOREM 1. *Let $f\in A^{1+\varepsilon}(D)$ for some $\varepsilon>0$ satisfy*

$$s_n(f) = 0 \quad \text{and} \quad s_{0n}(f) = 0 \tag{1}$$

for $n=1, 2, \cdots$. Then f is the zero function. Furthermore, for each positive integer n there exist two rational functions

$$q_n(z) = \sum_{k=-n}^{n} a_k z^k, \qquad q_{0n}(z) = \sum_{k=-n}^{n} a_{0k} z^k$$

with $a_0=a_{00}=0$ such that $s_m(q_n)=\delta_{m,n}$, $s_{0m}(q_n)=0$, $s_m(q_{0n})=0$ and $s_{0m}(q_{0n})=\delta_{n,m}$ for all m, $n=1, 2, \cdots$.

PROOF. Since f is holomorphic in D, we write $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ with

$$a_0 = \frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{dz}{z} = \lim_{n\to\infty} s_n(f) = 0.$$

Let $g(z)=\sum_{n=1}^{\infty}(a_n+a_{-n})z^n$. Then $g\in A^{1+\varepsilon}(U)$ and $s_n(g)=s_n(f)=0$ for all $n=1,2,\cdots$. Hence, we can conclude from a uniqueness theorem in **[1]** that $a_n+a_{-n}=0$ for all n. Similarly, we also consider

$$h(z) = \sum_{n=1}^{\infty}\left(a_n r_0^n + a_{-n}\frac{1}{r_0^n}\right)z^n,$$

and conclude that $s_n(h)=s_{0n}(f)$, $n=1,2,\cdots$, and hence that $a_n r_0^n + a_{-n}r_0^{-n}=0$ for all n. Since $0<r_0<1$, it is clear that $a_n=0$ for all n.

Next, we prove the existence of q_n. The proof of the existence of q_{0n} is similar. Since $s_m(q_n)=s_{0m}(q_n)=0$ for all $m>n$, we need only consider the following two systems of n equations:

$$\begin{aligned}
s_1(q_n) &= (a_1+a_{-1})+\cdots+(a_n+a_{-n}) = 0\\
s_2(q_n) &= (a_2+a_{-2})+(a_4+a_{-4})+\cdots = 0\\
&\cdot\\
&\cdot\\
&\cdot\\
s_{n-1}(q_n) &= (a_{n-1}+a_{-(n-1)}) = 0\\
s_n(q_n) &= a_n+a_{-n} = 1;
\end{aligned}$$

$$\begin{aligned}
s_{01}(q_n) &= (a_1 r_0 + a_{-1}r_0^{-1})+\cdots+(a_n r_0^n + a_{-n}r_0^{-n}) = 0\\
s_{02}(q_n) &= (a_2 r_0^2 + a_{-2}r_0^{-2})+(a_4 r_0^4 + a_{-4}r_0^{-4})+\cdots = 0\\
&\cdot\\
&\cdot\\
&\cdot\\
s_{0,n-1}(q_n) &= (a_{n-1}r_0^{n-1} + a_{-(n-1)}r_0^{-(n-1)}) = 0\\
s_{0n}(q_n) &= a_n r_0^n + a_{-n}r_0^{-n} = 0.
\end{aligned}$$

Since the coefficient matrices for (a_k+a_{-k}) and $(a_k r_0^k + a_{-k}r_0^{-k})$ are nonsingular, there are unique solutions for (a_k+a_{-k}) and $(a_k r_0^k+a_{-k}r_0^{-k})$, and hence for a_k and a_{-k}, $k=1,\cdots,n$.

To establish our representation theorem, we first obtain explicit formulas for q_n and q_{0n}. Let $\mu(n)$ be the Möbius function of n:

$$\begin{aligned}
\mu(n) &= 1, && \text{if } n=1,\\
&= (-1)^k, && \text{if } n = q_1\cdots q_k,\\
&= 0, && \text{if } p^2 \mid n \text{ for some } p>1,
\end{aligned}$$

where $q_1,\cdots,q_k$ are distinct primes.

LEMMA 1. *For each* $n=1, 2, \cdots,$

$$q_n(z) = \sum_{j\mid n} \frac{\mu(n/j)}{r_0^{-j} - r_0^j}\left\{\left(\frac{z}{r_0}\right)^j - \left(\frac{z}{r_0}\right)^{-j}\right\} \tag{2}$$

and

$$q_{0n}(z) = \sum_{j\mid n} \frac{\mu(n/j)}{r_0^j - r_0^{-j}} \{z^j - z^{-j}\}. \tag{3}$$

PROOF. We observe that the means

$$s_n\left(\frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j}\right) = \begin{cases} 1, & \text{if } n \mid j, \\ 0, & \text{if } n \nmid j \end{cases}$$

and

$$s_{0n}\left(\frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j}\right) = 0 \quad \text{for all } n = 1, 2, \cdots.$$

Hence, by virtue of Theorem 1, we have

$$\frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j} = \sum_{n\mid j} q_n(z)$$

for $j=1, 2, \cdots$. We now use the Möbius inversion theorem (cf. [5]) to obtain (2). The proof of (3) is similar.

THEOREM 2. *Let* $f \in A^{1+\varepsilon}(D)$ *for some* $\varepsilon>0$. *Then the series*

$$\sum_{k=1}^{\infty} R_k(f)q_k(z) + \sum_{k=1}^{\infty} R_{0k}(f)q_{0k}(z) + s_\infty(f) \tag{4}$$

converges uniformly to f *on* $\bar{D}$ *and*

$$\left|f(z) - \sum_{k=1}^{m} R_k(f)q_k(z) - \sum_{k=1}^{m} R_{0k}(f)q_{0k}(z) - s_\infty(f)\right| = O\left(\frac{1}{m^\delta}\right)$$

uniformly on $\bar{D}$ *for any fixed* δ, $0<\delta<\varepsilon$.

The series (4) is now called the *Riemann series* of the function f in D (cf. [3]).

PROOF. For $r_0 \leqq |z| \leqq 1$, we have

$$|q_k(z)| \leqq \sum_{j\mid k} \frac{1 + r_0^{2j}}{1 - r_0^{2j}} \leqq \frac{2d(k)}{1 - r_0^2}$$

where $d(k)$ denotes the number of divisors of k. Using the well-known estimate $d(k)=O(k^{\varepsilon-\delta})$, where $0<\delta<\varepsilon$ (cf. [5]), and the fact that $R_k(f)=O(1/k^{1+\varepsilon})$ and $R_{0k}(f)=O(r_0^k/k^{1+\varepsilon})$, which follows from the assumptions on f (cf. [2]), we can conclude that the series (4) converges uniformly on $\bar{D}$ to some function h, holomorphic in D and continuous on $\bar{D}$. Furthermore, we have

$$\left|h(z)-\sum_{k=1}^{m} R_k(f)q_k(z)-\sum_{k=1}^{m} R_{0k}(f)q_{0k}(z)-s_\infty(f)\right| = O\left(\frac{1}{m^\delta}\right)$$

uniformly on $\bar{D}$. Now, we use Lemma 1 to estimate the Fourier coefficients of h: For $m>0$,

$$\begin{aligned} a_m(h) &= a_m\left[\sum_{k=1}^{\infty} R_k(f)\sum_{j\mid k}\frac{\mu(k/j)}{r_0^{-j}-r_0^j}\{(z/r_0)^j-(z/r_0)^{-j}\}\right.\\ &\qquad\left.+\sum_{k=1}^{\infty} R_{0k}(f)\sum_{j\mid k}\frac{\mu(k/j)}{r_0^j-r_0^{-j}}(z^j-z^{-j})+s_\infty(f)\right]\\ &= \frac{1}{1-r_0^{2m}}\sum_{k=1}^{\infty} R_{mk}(f)\mu(k)+\frac{r_0^m}{1-r_0^{2m}}\sum_{k=1}^{\infty} R_{0,mk}(f)\mu(k)=O\left(\frac{1}{m^{1+\varepsilon}}\right). \end{aligned}$$

Similarly, for $m<0$,

$$\begin{aligned} a_m(h) &= \frac{-r_0^{-2m}}{1-r_0^{-2m}}\sum_{k=1}^{\infty} R_{-mk}(f)\mu(k)+\frac{-r_0^{-m}}{1-r_0^{-2m}}\sum_{k=1}^{\infty} R_{0,-mk}(f)\mu(k)\\ &= O(r_0^{|m|}/|m|^{1+\varepsilon}). \end{aligned}$$

Hence, $h\in A^{1+\varepsilon}(D)$ and the means of h are

$$\begin{aligned} s_m(h) &= s_m\left[\sum_{k=1}^{\infty} R_k(f)q_k+\sum_{k=1}^{\infty} R_{0k}(f)q_{0k}+s_\infty(f)\right]\\ &= \sum_{k=1}^{\infty} R_k(f)\delta_{m,k}+s_\infty(f)=R_m(f)+s_\infty(f)=s_m(f), \end{aligned}$$

and similarly, $s_{0m}(h)=R_{0m}(f)+s_\infty(f)=s_{0m}(f)$, for all $m=1,2,\cdots$. Hence, $f=h$ by Theorem 1.

For each $n=1,2,\cdots$, let $p_n(z)=\sum_{k\mid n}\mu(n/k)z^k$ as in [3]. We have

THEOREM 3. *Let $f\in A^{1+\varepsilon}(D)$ for some $\varepsilon>0$. Then f is in $A^{1+\varepsilon}(U)$ if and only if for all $m\geq 1$*

$$R_{0m}(f)=\sum_{j=1}^{\infty} p_j(r_0^m)R_{mj}(f). \tag{5}$$

Here, it is clear that the series in (5) converges for every f in $A^{1+\varepsilon}(D)$.

PROOF. An easy calculation shows that

$$R_{0k}(p_j) = p_\alpha(r_0^k) \quad \text{if } j = \alpha k$$
$$= 0 \qquad \text{if } k \nmid j. \tag{6}$$

In [3], it is proved that if $f \in A^{1+\varepsilon}(U)$ then $f(z)=\sum_{k=1}^{\infty} R_k(f)p_k(z)+s_\infty(f)$ uniformly in $\bar{U}$. Hence, we have, by (6),

$$R_{0m}(f) = \sum_{j=1}^{\infty} R_{mj}(f) \sum_{\alpha | j} \mu\left(\frac{j}{\alpha}\right) r_0^{m\alpha}$$

which is (5). To prove the converse, we first prove the following identities for all k and n:

$$\sum_{j|k} p_j(r_0^{kn/j})\mu\left(\frac{k}{j}\right) = r_0^n \mu(k). \tag{7}$$

Indeed,

$$\sum_{j|k} p_j(r_0^{kn/j})\mu\left(\frac{k}{j}\right) = \sum_{j|k} \mu\left(\frac{k}{j}\right) \sum_{\alpha|j} \mu\left(\frac{j}{\alpha}\right) r_0^{\alpha kn/j}$$
$$= \sum_{j|k} \mu\left(\frac{k}{j}\right) \sum_{\alpha|j} \mu(\alpha) r_0^{kn/\alpha} = \sum_{\alpha|k} r_0^{kn/\alpha}\mu(\alpha) \sum_{j|(k/\alpha)} \mu\left(\frac{k}{j\alpha}\right),$$

so that (7) follows from the identity $\sum_{j|n} \mu(j)=\delta_{1,n}$.

From Theorem 2, we have

$$f(z) = \sum_{j=1}^{\infty} R_j(f)q_j(z) + \sum_{j=1}^{\infty} R_{0j}(f)q_{0j}(z) + s_\infty(f) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

It is clear that for each $n>0$,

$$a_{-n} = \sum_{m=1}^{\infty} \frac{\mu(m)}{r_0^n - r_0^{-n}} R_{nm}(f) r_0^n + \sum_{m=1}^{\infty} \frac{\mu(m) R_{0,mn}(f)}{r_0^n - r_0^{-n}}.$$

Since $R_k(f)=O(1/k^{1+\varepsilon})$, we obtain, by (5) and (7),

$$(r_0^n - r_0^{-n})a_{-n} = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} p_j(r_0^{nm}) R_{jmn}(f) - \sum_{m=1}^{\infty} r_0^n R_{nm}(f)\mu(m)$$
$$= \sum_{k=1}^{\infty} R_{kn}\left\{\sum_{j|k} p_j(r_0^{kn/j})\mu\left(\frac{k}{j}\right) - r_0^n \mu_k\right\} = 0.$$

3. **Final remark.** The results in this paper are generalizations of those studied in ([1], [3]). Recently, Patil ([6], [7]) has given an explicit representation of an H^p function in terms of its boundary values on a small subset S of the unit circle. It is, therefore, also interesting to know whether or not just the arithmetic means of the values of a function $f \in A^{1+\varepsilon}(U)$ at

points "equally spaced" on S would uniquely determine f, and if so, whether or not an "explicit" formula for recapturing f from these means could be given. If S is an arc, some results have been recently obtained in [**4**].

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