STIELTJES DIFFERENTIAL-BOUNDARY OPERATORS

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ABSTRACT. The differential-boundary system S :

$$\begin{aligned} Ly &= (y + H(t)[Cy(0) + Dy(1)] + H_1(t)\psi)' + P(t)y. \\ Ay(0) + By(1) + \int_0^1 dK(t)y(t) &= 0, \quad \int_0^1 dK_1(t)y(t) = 0, \end{aligned}$$

is discussed when set in the space $\mathcal{L}_n^p[0, 1]$. The density of the domain of L is discussed, and the adjoint or dual operator is derived. A discussion of selfadjoint systems follows. Necessary and sufficient conditions for $T=(1/i)L$ to be selfadjoint in $\mathcal{L}_n^2[0, 1]$ are given.

1. Introduction. Differential-boundary operators with discontinuities at a finite number of points were thoroughly discussed in [6]. Stieltjes differential-boundary operators were recently encountered by Brown and Krall [1] as dual operators when extending the work of Bryan [2]. In addition Bryan [3] had earlier derived Green's formula as well as other results for Stieltjes differential-boundary operators in the case where $H_1=0$.

In addition we should mention the work of Vejvoda and Tvrdy [7] which presented a lengthy discussion of the algebraic-measure theoretic properties required of S , and finally, the work by Halanay and Moro [4], which discussed the existence and uniqueness of solutions for S .

Our point of view differs somewhat from those just mentioned written by others in that we will consider S as generated by a differential-boundary operator in $\mathcal{L}_n^p[0, 1]$, $1 \leq p < \infty$, then derive the dual operator in $\mathcal{L}_n^q[0, 1]$, $1/p + 1/q = 1$.

Specifically, in $\mathcal{L}_n^p[0, 1]$ we wish to define the operator L in the following way: (1) Let A and B be $m \times n$ matrix constants with $m \leq 2n$ such that $\text{rank}(AB) = m$. Let C and D be $(2n-m) \times n$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular.

(2) Let K be an $m \times n$ matrix valued function of bounded variation such that, when considered as generating a matrix valued measure, $dK(0) = A$, $dK(1) = B$.

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(3) Let K_1 be an $r \times n$ matrix valued function of bounded variation such that, when considered as generating a matrix valued measure, $dK_1(0)=0$, $dK_1(1)=0$.

Thus the matrix $(\overset{K}{K}_1)$ generates an $(m+r) \times n$ matrix valued measure such that the rank of

$$\begin{pmatrix} dK(0) & dK(1) \\ dK_1(0) & dK_1(1) \end{pmatrix}$$

is m , the portion generating this rank lying in the first m rows. Any arbitrary boundary condition of the form $\int_0^1 dv(t)y(t)=0$ can be expressed in such a manner.

(4) Let H and H_1 be $n \times (2n-m)$ and $n \times s$ matrix valued functions of bounded variation.

(5) Let P be a continuous $n \times n$ matrix.

1.1. DEFINITION. We denote by D those elements y in $\mathcal{L}_n^p[0, 1]$ satisfying

(a) for each y there exists an $s \times 1$ matrix valued constant ψ such that $y + H[Cy(0) + Dy(1)] + H_1\psi$ is absolutely continuous.

(b) $ly = (y + H[Cy(0) + Dy(1)] + H_1\psi)' + Py$ exists a.e. and is in $\mathcal{L}_n^p[0, 1]$.

(c) $Ay(0) + By(1) + \int_0^1 dK(t)y(t) = 0$, $\int_0^1 dK_1(t)y(t) = 0$.

We define the differential-boundary operator L by letting $Ly = ly$ for all y in D .

Before we can proceed further we must know whether or not D is dense in $\mathcal{L}_n^p[0, 1]$. The adjoint or dual operator will not exist in the ordinary sense unless it is. To decide this question we introduce the notion of the kernel of a measure.

1.2 DEFINITION. Let the measure ν be generated by the matrix of bounded variation $(\overset{K}{K}_1)$ so that

$$\int_0^1 dv(t)y(t) = \begin{pmatrix} Ay(0) + By(1) + \int_0^1 dK(t)y(t) \\ \int_0^1 dK_1(t)y(t) \end{pmatrix}.$$

Let T be a set which is everywhere dense in $[0, 1]$. For fixed t the kernel of $\nu^*(t)$ is the set of vectors v satisfying $\nu^*(t)v=0$.¹ By the kernel $K_{\nu^*}^T$ of ν^* with respect to T we mean $\bigcap_{t \in T} \ker \nu^*[0, t]$.

It can be shown that if $1 \in T$, then $K_{\nu^*}^T$ is independent of the particular choice of T when compared with kernels for other dense sets [1].

¹ The * indicates the matrix conjugate transpose.

1.2. THEOREM. Let $v^* = v_s^* + v_c^*$, where $v_s^* \perp v_c^*$, v_s^* is singular with respect to Lebesgue measure, and v_c^* is absolutely continuous with respect to Lebesgue measure. D is dense in $\mathcal{L}_n^q[0, 1]$ if and only if $K_{v_s^*}^T \subset K_{v_c^*}^T$.

The proof is found in [1]. In our case we need only apply it to those elements in D for which $\varphi=0$.

We shall assume throughout the remainder of this paper that the boundary conditions are acceptable: that D is dense in $\mathcal{L}_n^q[0, 1]$. This implies that the dual operator does exist in $\mathcal{L}_n^q[0, 1]$, $1/p+1/q=1$.

2. The dual operator. Let us denote the inverse of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by

$$\begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix}.$$

i.e.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} = \begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{2n}.$$

Componentwise these products imply

$$\begin{aligned} -A\tilde{A}^* + B\tilde{B}^* &= I, & -A\tilde{C}^* + B\tilde{D}^* &= 0, \\ -C\tilde{A}^* + D\tilde{B}^* &= 0, & -C\tilde{C}^* + D\tilde{D}^* &= I, \end{aligned}$$

and

$$\begin{aligned} \tilde{A}^*A + \tilde{C}^*C &= -I, & \tilde{A}^*B + \tilde{C}^*D &= 0, \\ \tilde{B}^*A + \tilde{D}^*C &= 0, & \tilde{B}^*B + \tilde{D}^*D &= I, \end{aligned}$$

it being understood that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are $m \times n$, $m \times n$, $(2n-m) \times n$ and $(2n-m) \times n$ matrices.

2.1. DEFINITION. We denote by D^+ those elements z in $\mathcal{L}_n^q[0, 1]$ satisfying

(a) for each z there exists an $r \times 1$ matrix valued constant ϕ such that $z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi$ is absolutely continuous.

(b) $l^+z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + P^*z$ exists a.e. and is in $\mathcal{L}_n^q[0, 1]$.

(c) $\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*(t)z(t) = 0$, $\int_0^1 dH_1^*(t)z(t) = 0$.

We define the differential-boundary operator L^+ by letting $L^+z = l^+z$ for all z in D^+ .

Our aim is to show that the dual of the operator L is L^+ . In order to do so, however, it is necessary to have two forms of Green's formula available to us.

Let D_0 denote those elements y in $\mathcal{L}_n^p[0, 1]$ satisfying:

(a) For each y there exists an $s \times 1$ matrix valued constant ψ such that $y + H[Cy(0) + Dy(1)] + H_1\psi$ is absolutely continuous.

(b) $ly = (y + H[Cy(0) + Dy(1)] + H_1\psi)' + Py$ exists a.e. and is in $\mathcal{L}_n^p[0, 1]$.

Let D_0^+ denote those elements z in $\mathcal{L}_n^q[0, 1]$ satisfying:

(a) For each z there exists an $r \times 1$ matrix valued constant ϕ such that $z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi$ is absolutely continuous.

(b) $l^+z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + P^*z$ exists a.e. and is in $\mathcal{L}_n^q[0, 1]$.

Let D_0^{++} denote those elements z in $\mathcal{L}_n^q[0, 1]$ satisfying:

(a) For each z there exist an $r \times 1$ matrix valued constant ϕ and an $m \times 1$ matrix valued constant ϕ_1 such that $z + K^*\phi_1 + K_1^*\phi$ is absolutely continuous.

(b) $l^{++}z = -(z + K^*\phi_1 + K_1^*\phi)' + P^*z$ exists a.e. and is in $\mathcal{L}_n^q[0, 1]$.

The following two results can be easily derived (see Bryan [3], Vejvoda and Tvrdy [7]).

2.2. THEOREM (GREEN'S FORMULA). *Let y be in D_0 , z be in D_0^+ . Then*

$$\begin{aligned} & \int_0^1 [z^*(ly) - (l^+z)^*y] dt \\ &= \left[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z \right]^* [Cy(0) + Dy(1)] \\ &+ [\tilde{A}z(0) + \tilde{B}z(1)]^* \left[Ay(0) + By(1) + \int_0^1 dKy \right] \\ &+ \phi^* \left[\int_0^1 dK_1y \right] + \left[\int_0^1 dH_1^*z \right]^* \psi. \end{aligned}$$

2.3. THEOREM. *Let y be in D_0 , z be in D_0^{++} . Then*

$$\begin{aligned} & \int_0^1 [z^*(ly) - (l^{++}z)^*y] dt \\ &= \left[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z \right]^* [Cy(0) + Dy(1)] \\ &+ [\tilde{A}z(0) + \tilde{B}z(1)]^* \left[Ay(0) + By(1) + \int_0^1 dKy \right] \\ &+ \phi^* \left[\int_0^1 dK_1y \right] + \left[\int_0^1 dH_1^*z \right]^* \psi \\ &+ [\phi_1 - (\tilde{A}z(0) + \tilde{B}z(1))]^* \left[\int_0^1 dKy \right]. \end{aligned}$$

Remarks are now in order concerning the form of these operators.

We shall say that an operator of the form $(y + H[Cy(0) + Dy(1)])' + Py$ is endpoint restricted because of the nature of the adjusting term $H[Cy(0) + Dy(1)]$. An operator of the form $(y + H_1\psi)' + Py$ is free since ψ is undetermined by specific points. Operators of the form given by l are mixed in form. There exists a multitude of other mixed operators: For instance, the operators discussed in [6] are of the form

$$\left(y + \sum_{i=1}^m H_i \left[\sum_{j=1}^m (C_{ij}y(a_{j-1}+) + D_{ij}y(a_j-)) \right] \right)' + Py$$

where H_i , $i=1, \dots, n$, are absolutely continuous except at the discrete set of points $\{a_j\}_{j=1}^m$ are mixed.

2.4. THEOREM. *In dual of L in $\mathcal{L}_n^p[0, 1]$ is given by L^+ in $\mathcal{L}_n^q[0, 1]$.*

PROOF. Let L_0 be the restriction of L to those elements in D which vanish at 0 and 1 and for which $\psi=0$. This operator was discussed in [1] where its dual was shown to have the form

$$L_0^*z = l^{++}z = -(z + K^*\phi_1 + K_1^*\phi)' + P^*z.$$

Since $L_0 \subset L$, $L^* \subset L_0^*$, and $L^*z = -(z + K^*\phi_1 + K_1^*\phi)' + P^*z$. An application of Theorem 2.3 for arbitrary y in D shows that

$$\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z = 0, \quad \int_0^1 dH_1^*z = 0,$$

and

$$\phi_1 = \tilde{A}z(0) + \tilde{B}z(1),$$

since the coefficients of these terms are unrestricted. This shows that $L^* \subset L^+$. The reverse inclusion is a trivial consequence of Green's formula.

2.4. COROLLARY. *Let y be in D , z be in D^+ , then*

$$\int_0^1 [z^*(Ly) - (L^+z)^*y] dt = 0.$$

3. Selfadjoint systems. *Let us restrict our considerations to $\mathcal{L}_n^2[0, 1]$. In order to find selfadjoint differential-boundary operators it is convenient to replace the operator L by*

$$Ty = (1/i)(y + H[Cy(0) + Dy(1)] + H_1\psi)' + Qy$$

where $(1/i)P=Q$. T^* is then given by

$$T^*z = (1/i)(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + Q^*z.$$

Green's formula becomes

$$\begin{aligned} \int_0^1 [z^*(Ty) - (T^*z)^*y] dt \\ = (1/i) \left\{ \left[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z \right]^* [Cy(0) + Dy(1)] \right. \\ \left. + [\tilde{A}z(0) + \tilde{B}z(1)]^* \left[Ay(0) + By(1) + \int_0^1 dKy \right] \right. \\ \left. + \phi^* \left[\int_0^1 dK_1y \right] + \left[\int_0^1 dH_1^*z \right]^* \psi \right\}. \end{aligned}$$

We then conclude immediately:

3.1. THEOREM. T is selfadjoint in $\mathcal{L}_n^2[0, 1]$ if and only if

- (1) $Q=Q^*$.
- (2) $m=n, r=s$.
- (3) $K=[BD^*-AC^*]H^*$ a.e.
- (4) $AA^*=BB^*$.
- (5) $H[CC^*-DD^*]=0$ a.e.
- (6) $K_1=MH_1^*$, where M is a nonsingular $r \times r$ matrix.

The proof is quite similar to that found in [6]. Let T be selfadjoint. Then y absolutely continuous and vanishing at 0 and 1 (so $\psi=0, \phi=0$ also) shows $Q=Q^*$. A comparison of the number of boundary conditions and relative sizes of H_1 and K_1 shows $m=n, r=s$. If the elements y vanish at 0 and 1 but are not necessarily absolutely continuous, then $H_1\psi=K_1^*\phi$, which in turn implies $K_1=MH_1^*$.

At this point we see that

$$H[Cy(0) + Dy(1)] = K_1^*[\tilde{A}y(0) + \tilde{B}y(1)].$$

Since y may or may not vanish at 0 or 1, we must have $HC=K^*\tilde{A}$, $HD=K^*\tilde{B}$. Multiplying by C^* and D^* and subtracting, we find $H[CC^*-DD^*]=0$. Multiplying by A^* and B^* and subtracting, we find $K=[BD^*-AC^*]H^*$.

Finally by comparing the first boundary conditions, we find $A=E\tilde{C}$, $B=E\tilde{D}$, $K=E\tilde{H}^*$. The last we have already shown to be true. Multiplying the first by A^* , the second by B^* and subtracting, we find $AA^*=BB^*$.

The converse is similar to that found in [6].

4. **Remarks.** If the spectrum of the operator T is discrete, then an eigenfunction expansion like that found in [6] follows immediately. The solution of the problem $Ty=\lambda y$ has not been thoroughly enough explored to say that this is definitely the case, however. The Hildebrandt

matrix style differentio-Stieltjes-integral equation [5] which is needed seems to possess a fundamental matrix, but its exact nature as well as its analyticity in λ has not been suitably verified. We plan to follow this paper with such a verification.

The derivation of suitable nonselfadjoint eigenfunction expansions also leads to a differentio-Stieltjes-integral equation, which in the context of the present paper involves a 5 dimensional matrix system. The same, as well as other, problems are encountered. Again we hope to present a solution in the near future.

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