ESCAPE FROM THE UNIT INTERVAL UNDER THE TRANSFORMATION $x \mapsto \lambda x(1-x)$

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ABSTRACT. For $\lambda > 4$ the transformation $x \mapsto \lambda x(1-x)$ maps the unit interval into itself except for an interval of x values centered at $\frac{1}{2}$ that "escapes". It is shown that almost all of the unit interval eventually escapes if the transformation is iterated. An easy example is then given for which the theorem fails, and the question is raised for exactly what class of functions the theorem holds.

Stein and Ulam [1] and Metropolis, Stein and Stein [2] have studied the iterative properties on the unit interval of the transformation $x \mapsto \lambda x(1-x)$ for $3 < \lambda \leq 4$.

In a biological study (McNaughton and Henry [3]), the same transformation came up as the result of changing Verhulst's differential equation of population growth $dp/dt = mp - np^2$ to a difference equation as is appropriate for populations with synchronous life cycles. Moreover, the parameter range $\lambda > 4$ became interesting so that the iteration can escape the unit interval. Biologically, escape means extinction.

Though for any $\lambda > 4$ it is clear that some x leave the unit interval under $x \mapsto \lambda x(1-x)$ (called f from now on), it is not obvious for how "fat" a set of x iteration of f will *eventually* cause escape. Our purpose is to prove: *For almost all $x \in (0, 1)$ some iterate of x under f ($\lambda > 4$) is not in $(0, 1)$* (almost all in the sense of Lebesgue measure).

Proof of theorem. For $\lambda > 4$ let I be the closed subinterval of $(0, 1)$ of values x for which $f(x) \geq 1$. With

$$\delta = \left(\frac{1}{4} - \frac{1}{\lambda}\right)^{1/2}, \quad I = \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right].$$

Let U stand for the open interval $(0, 1)$. For $x \in U$ let $k(x) \leq \infty$ be the number of steps to escape starting from x . $k(x) \geq 1$ and $k(x) = 1$ iff

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$x \in I$. If $k(x) = r+1$ then with f^{or} standing for the r -fold iterate of f and f^{o-r} its complete r -fold iterated inverse, $f^{or}(x) \in I$ so that $x \in f^{o-r}(I)$. Clearly

$$(1) \quad U = I + f^{-1}(U)$$

(+ indicates disjoint union). r inverse iterations of (1) result in

$$U = I + f^{-1}(I) + \cdots + f^{o-r}(I) + f^{o-(r+1)}(U).$$

Now $f^{o-r}(I)$ (resp. $f^{o-r}(U)$) is the disjoint union of 2^{r-1} pairs of closed (resp. open) intervals, the members of each pair equal and symmetrically located above and below $x = \frac{1}{2}$. $A = I + f^{-1}(I) + f^{o-2}(I) + \cdots$ is the set of eventual extinction. We shall prove the Lebesgue measure $|A|$ of A is unity by showing that

$$|f^{o-(n+1)}(U)| = 1 - |I + f^{-1}(I) + \cdots + f^{o-n}(I)| \rightarrow 0.$$

Put $f^{o-n}(I) = I_n$, $f^{o-n}(U) = U_n$. Numbering components left to right we have

$$I_n = I_{n1} + \cdots + I_{n2^n}, \quad U_n = U_{n1} + \cdots + U_{n2^n}.$$

Here is a lemma proved up ahead: *For all n, k , $|U_{nk}| \leq c^n$ where $c = \max(4/\lambda, 1/2)$.*

If I_{nk} (resp. U_{nk}) is a component of $f^{-1}(I_{n-1,k_1})$ (resp. $f^{-1}(U_{n-1,k_1})$) then

$$(2) \quad \frac{|I_{nk}|}{|U_{nk}|} = \frac{|I_{n-1,k_1}|/s_{nk}}{|U_{n-1,k_1}|/t_{nk}} = \frac{t_{nk}}{s_{nk}} \frac{|I_{n-1,k_1}|}{|U_{n-1,k_1}|},$$

where $s_{nk} = |\text{slope}|$ of f at center of I_{nk} ; $t_{nk} = |\text{slope}|$ of f at center of U_{nk} . But $(d^2/dx^2)f(x) = -2\lambda = \text{rate of change of slope of } f$, and the centers of I_{nk} and U_{nk} are $\leq |U_{nk}|/2$ apart since $I_{nk} \subset U_{nk}$, so $|s_{nk} - t_{nk}| \leq \lambda |U_{nk}|$. The minimum $|\text{slope}|$ of the curve within the closed unit square is $2\delta\lambda$. Hence

$$(3) \quad \left| 1 - \frac{t_{nk}}{s_{nk}} \right| = \frac{1}{s_{nk}} |s_{nk} - t_{nk}| \leq \frac{\lambda}{2\delta\lambda} |U_{nk}| \leq \frac{1}{2\delta} c^n$$

by the lemma.

Iterating equation (2) $(n-1)$ times gives

$$\frac{|I_{nk}|}{|U_{nk}|} = \frac{t_{nk}}{s_{nk}} \cdot \frac{t_{n-1,k_1}}{s_{n-1,k_1}} \cdots \frac{t_{1,k_{n-1}}}{s_{1,k_{n-1}}} \frac{|I|}{|U|}$$

where, in an obvious extension of the terminology, U_{n-j,k_j} is one component of $f^{-1}(U_{n-j-1,k_{j+1}})$; t_{n-j,k_j} is $|\text{slope}|$ of f at center of U_{n-j,k_j} . Or,

$$\frac{|I_{nk}|}{|U_{nk}|} = \frac{|I|}{|U|} \cdot \prod_{j=1}^n \left\{ 1 - \left[1 - \frac{t_{j,k_{n-j}}}{s_{j,k_{n-j}}} \right] \right\}.$$

The product on the right converges absolutely as $n \rightarrow \infty$, hence stays bounded away from zero, provided the series $\sum_{j=1}^{\infty} |1 - (t_{jk_j}/s_{jk_j})|$ converges. But that series is by (3) dominated by the convergent geometric series $\sum_{j=1}^{\infty} c^j$. We may therefore conclude $|I_{nk}|/|U_{nk}| \geq \varepsilon > 0$ uniformly for all n, k , so also $|I_n|/|U_n| = |f^{o-n}(I)|/|f^{o-n}(U)| \geq \varepsilon$ uniformly for all n . It is now clear that $|f^{o-n}(U)| \rightarrow 0$ geometrically, and the proof is complete.

Proof of lemma. For any λ , $|U_{11}| = |U_{12}| < \frac{1}{2} \leq c$. Assume inductively that $|U_{ji}| \leq c^j$ for $1 < j \leq n-1$ and consider U_{nk} , with midpoint y_0 . If U_{nk} is so positioned that $|f'(y_0)| \geq 1/c$ there is no more to prove.

Now $|f'(y_0)| = 2\lambda|y_0 - \frac{1}{2}|$. By symmetry we may assume $U_{nk} \subset (\frac{1}{2}, 1)$ by which further analysis is required only for $y_0 < \frac{5}{8}$. If λ is large enough that $\delta \geq \frac{1}{8}$, then $y_0 \geq \frac{5}{8}$, so we need consider further only $\delta < \frac{1}{8}$, or $2\delta = ((\lambda-4)/\lambda)^{1/2} < \frac{1}{4} \Rightarrow \lambda < 64/15$. In particular, the proof is complete for $\lambda > 8$, so from now on we use $4/\lambda$ instead of c .

Call x_0 the distance of the right end of U_{nk} from the right end of I . Since $x_0 < 2[y_0 - (\frac{1}{2} + \delta)]$ and $y_0 < \frac{5}{8}$, $x_0 < 2(\frac{1}{8} - \delta)$, or $x_0 + 2\delta < \frac{1}{4}$. Put

$$\begin{aligned} x_1 &= 1 - f(\tfrac{1}{2} + \delta + x_0) = 1 - \text{left endpoint } U_{n-1, k_1}, \\ x_2 &= f(x_1), \quad \dots, \quad x_r = f(x_{r-1}). \end{aligned}$$

r will be the number of "back-steps" needed to make the product of slopes large enough to complete the induction.

Define $M'_0, M'_1, \dots, M'_{r-1}$ by $x_1 = M'_0 x_0, x_2 = M'_1 x_1, \dots, x_r = M'_{r-1} x_{r-1}$. M'_j is a secant slope of the curve. Define $M_0, M_1, \dots, M_{r-1}$ by

$$\begin{aligned} |U_{n-1, k_1}| &= M_0 |U_{nk}|, \\ |U_{n-2, k_2}| &= M_1 |U_{n-1, k_1}|, \\ &\vdots \\ |U_{n-r, k_r}| &= M_{r-1} |U_{n-(r-1), k_{r-1}}|. \end{aligned}$$

Here $U_{n-1, k_1} = f(U_{nk})$, $U_{n-2, k_2} = f(U_{n-1, k_1})$, etc.

Now $M'_1 = \lambda(1 - x_1), \dots, M'_r = \lambda(1 - x_r)$, $M_0 > M'_0$, and provided $x_0, x_1, \dots, x_r < \frac{1}{2}$,

$$(4) \quad M_1 > \lambda(1 - 2x_1), \quad \dots, \quad M_r > \lambda(1 - 2x_r).$$

To complete the induction requires some one of the inequalities $M_0 \geq \lambda/4$, $M_1 \geq (\lambda/4)^2, \dots, M_r \geq (\lambda/4)^{r+1}$. In view of the inequalities

(4), any of the following will suffice:

$$\begin{aligned} M'_0 &\geq \lambda/4, \\ M'_0 \lambda (1 - 2x_1) &\geq (\lambda/4)^2, \\ M'_0 \lambda^2 (1 - 2x_1)(1 - 2x_2) &\geq (\lambda/4)^3, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

For $M'_0 = \lambda(x_0 + 2\delta) \geq \lambda/4$, the induction is already complete. That happens when $x_0 + 2\delta \geq \frac{1}{4}$, or $x_0 \geq \frac{1}{4} - 2\delta$.

Next, assume $x_0 < \frac{1}{4} - 2\delta$. Then

$$\begin{aligned} x_1 &= \lambda(x_0 + 2\delta)x_0 < \lambda(\tfrac{1}{4} - 2\delta + 2\delta)(\tfrac{1}{4} - 2\delta) = \tfrac{1}{4}\lambda(\tfrac{1}{4} - 2\delta) \\ &\Rightarrow M_1 > \lambda(1 - 2x_1) > \lambda[1 - \tfrac{1}{2}\lambda(\tfrac{1}{4} - 2\delta)] > \tfrac{1}{2}\lambda \\ &\Rightarrow M_0 M_1 \geq (\tfrac{1}{4}\lambda)^2 \end{aligned}$$

provided $M'_0 \geq \lambda/8$ or $M'_0/2\lambda = \frac{1}{2}x_0 + \delta \geq \frac{1}{16}$, completing induction for such x_0 .

Assume next that

$$(5) \quad \tfrac{1}{2}x_0 + \delta < \tfrac{1}{16} \Rightarrow x_0 < \tfrac{1}{8} - 2\delta.$$

Given (5), we have by (4):

$$(6) \quad x_1 < \tfrac{1}{8}\lambda(\tfrac{1}{8} - 2\delta) \Rightarrow M_1 > \lambda[1 - \tfrac{1}{4}\lambda(\tfrac{1}{8} - 2\delta)].$$

It is quickly verified from the dependence of δ on λ , using $\lambda < 64/15$, that $\tfrac{1}{4}\lambda(\tfrac{1}{8} - 2\delta) < \tfrac{1}{8}$. Hence by (6), $M_1 > 7\lambda/8$. That implies $M'_0 M_1 \geq (\lambda/4)^2$ whenever $M'_0 = (x_0 + 2\delta)\lambda \geq \lambda/14$. In particular, the induction is complete for $2\delta \geq 1/14$.

If now $x_0, x_1, \dots, x_r \leq \frac{1}{4}$, then $M_r \geq \lambda/2$ and $M_{r-1}, M_{r-2}, \dots, M_1 \geq \frac{3}{4}\lambda$. If further $M'_0 \geq \frac{8}{3}(\frac{1}{3})^{r-1}$, then

$$(7) \quad M_0 M_1 \cdots M_r \geq \tfrac{1}{8}\lambda(\tfrac{1}{3})^{r-1} \lambda^r (\tfrac{3}{4})^{r-1} \cdot \tfrac{1}{2} = \lambda^{r+1}/4^{r+1}.$$

The subsequent work is to verify this inequality.

We are now at $M'_0 < \lambda/14$. Fix r so that $(\frac{1}{3})^{r-1} \frac{1}{8} \lambda \leq M'_0 < (\frac{1}{3})^{r-2} \frac{1}{8} \lambda$ ($r \geq 2$). Then

$$\begin{aligned} x_j &= x_1 M'_1 M'_2 \cdots M'_{j-1} \quad (2 \leq j \leq r) \\ &\leq x_1 \lambda^{j-1} = x_0 M'_0 \lambda^{j-1}. \end{aligned}$$

By choice of r ,

$$\begin{aligned} (8) \quad x_0 &= (M'_0 - 2\delta\lambda)/\lambda < M'_0/\lambda < (\tfrac{1}{3})^{r-2} \cdot \tfrac{1}{8} \\ &\Rightarrow x_0 M'_0 < (\tfrac{1}{3})^{2(r-2)} \cdot \tfrac{1}{8} \lambda \cdot \tfrac{1}{8} \\ &\Rightarrow x_j < \lambda^j/8 \cdot 1/9^{r-2}. \end{aligned}$$

In the particular case $r=2$, $M'_0 < \lambda/14$. That implies

$$x_0 M_0 < \frac{M'_0}{\lambda} M'_0 < \frac{\lambda}{14^2} \Rightarrow x_2 \leq x_0 M'_0 \lambda < \frac{\lambda^2}{14^2} < \frac{1}{4},$$

verifying (7) for $r=2$. For $r=3$, by (8) $x_j < \lambda^j/8^2 \cdot \frac{1}{9} < \frac{1}{4}$ ($j=2, 3$), showing that (7) holds for $r=3$. For $r>3$, by (8),

$$x_2 < \frac{\lambda^2}{64} \cdot \frac{1}{9} < \frac{1}{4}, \quad x_j < \frac{\lambda^3}{64 \cdot 9} \cdot \frac{\lambda^{j-3}}{9^{r-2}} < \frac{1}{4} \quad (j = 3, \dots, r).$$

The induction is now complete in all cases: the lemma is proved.

Where does the theorem fail? It is easy to find concave functions on $(0, 1)$ for which almost sure escape fails, e.g.

$$h(x) = \begin{cases} 9/8 - |x - 1/2|, & (|x - 1/2| \leq 3/8) \\ 3 - 6|x - 1/2|, & (|x - 1/2| \geq 3/8) \end{cases}.$$

The question is open exactly *where* it fails. The methods used here do not get much further than the parabolic case.

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