A CLASS OF PURE SUBGROUPS OF COMPLETELY DECOMPOSABLE ABELIAN GROUPS

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ABSTRACT. Direct sum decompositions of the class of pure subgroups of finite rank completely decomposable torsion free abelian groups with typesets of cardinality at most 4 are considered. In certain cases, the indecomposable groups are classified, resulting in new proofs of several theorems by T. B. Cruddis.

The class $\mathcal{R}$ of pure subgroups of completely decomposable torsion free abelian groups of finite rank is considered by Butler [2]. Many of the examples of pathological direct sum decompositions of finite rank torsion free abelian groups occur in the class $\mathcal{R}$. On the other hand, homogeneous $\mathcal{R}$ -groups are completely decomposable and if A is an $\mathcal{R}$ -group, then $\text{typeset}(A)$ is finite and closed under intersection of types.

This note considers $\mathcal{R}$ -groups with typesets of cardinality at most 4. Motivation is provided by the curious results of Cruddis [3]. An easier and more illuminating proof of these results is a consequence of the following theorems.

Let $\Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ be a set of types with $\tau_0 = \tau_1 \cap \tau_2 \cap \tau_3$; let $\mathcal{R}_\Lambda$ be the class of $\mathcal{R}$ -groups with $\text{typeset} \subseteq \Lambda$; let $\sum_\Lambda$ be the sum of types in Λ ; and let $\sum_A$ be the sum of types in $\text{typeset}(A)$. Define τ_∞ to be the type represented by (∞) and, for p a prime, let τ_p be the type represented by (n_q) where $n_p = 0$ and $n_q = \infty$ if $p \neq q$.

THEOREM 1. *Suppose that A is an $\mathcal{R}_\Lambda$ -group and that $\text{typeset}(A)$ is a proper subset of Λ .*

- (a) *If A is indecomposable, then $\text{rank } A \leq 2$.*
- (b) *If $\sum_A = \tau_\infty$, then A is completely decomposable.*
- (c) *If $\sum_A = \tau_p$, for some prime p , then A has a unique decomposition (up to isomorphism) into indecomposable summands.*

THEOREM 2. *Suppose that A is an $\mathcal{R}_\Lambda$ -group with $\text{typeset}(A) = \Lambda$ and that $\sum_\Lambda = \tau_\infty$.*

- (a) *If A is indecomposable, then $\text{rank } A \leq 3$.*

Received by the editors February 7, 1973.

AMS (MOS) subject classifications (1970). Primary 20K15.

Key words and phrases. Torsion free abelian groups of finite rank, completely decomposable groups.

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(b) *There is a completely decomposable $\mathcal{R}_\Lambda$ -group H such that $A \oplus H = G_1 \oplus \cdots \oplus G_k$ and $\text{rank}(G_i) \leq 2$ for $1 \leq i \leq k$.*

Corollary 1.2 includes an explicit characterization of all indecomposable $\mathcal{R}_\Lambda$ -groups, where $\sum \Lambda = \tau_\infty$.

Let $\pi = \{p, q, r\}$ be a set of distinct primes, let $\mathcal{C}_3$ be the class of $\mathcal{R}$ -groups divisible by all primes not in π , and let $\Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ where $\tau_0 = \tau_p \cap \tau_q \cap \tau_r$, $\tau_1 = \tau_p \cap \tau_q$, $\tau_2 = \tau_p \cap \tau_r$ and $\tau_3 = \tau_q \cap \tau_r$.

COROLLARY 3 (CRUDDIS). *Let A be a $\mathcal{C}_3$ -group.*

(a) *$A = H \oplus G$ where H is completely decomposable and G is an $\mathcal{R}_\Lambda$ -group.*

(b) *Any two decompositions of A into indecomposable summands with rank ≤ 2 are equivalent.*

(c) *$K_0(\mathcal{C}_3)$ is a free abelian group with $S = \{[A] \mid A \text{ is indecomposable and } \text{rank } A \leq 2\}$ as a basis ($K_0(\mathcal{C}_3)$ is the Grothendieck group of $\mathcal{C}_3$, modulo direct sums, and $[A]$ is the element of $K_0(\mathcal{C}_3)$ determined by A).*

0. Preliminaries and notation. The basic reference for the elementary notions of torsion free abelian groups is Fuchs [4].

A *height sequence* is a sequence (n_p) of extended integers indexed by the set of primes of $\mathbb{Z}$, such that $0 \leq n_p \leq \infty$ for each prime p . Two height sequences (n_p) and (m_p) are *equivalent* if $n_p = m_p$ for all but a finite number of primes p with $n_p < \infty$ and $m_p < \infty$. A *type* is an equivalence class of height sequences. The natural ordering of height sequences induces an ordering on the set of all types (we write $\tau_1 \cap \tau_2$ for $\min\{\tau_1, \tau_2\}$) and addition of height sequences (with the convention that $\infty + k = \infty$) induces addition of types.

Assume that A is a torsion free abelian group, that $0 \neq a \in A$ and that p is a prime. Then $h_p(a)$ is the largest integer n with $a \in p^n A \setminus p^{n+1} A$ and ∞ if no such n exists. Define $h_A(a)$ to be the height sequence $(h_p(a))$; the *type* $T_A(a)$ of a in A is the equivalent class of $h_p(a)$. Note that

$$T_A(a_1 + \cdots + a_n) \geq T_A(a_1) \cap \cdots \cap T_A(a_n)$$

and if $A = A_1 \oplus \cdots \oplus A_n$ with $a_i \in A_i$, then

$$T_A(a_1 + \cdots + a_n) = T_{A_1}(a_1) \cap \cdots \cap T_{A_n}(a_n).$$

Define $\text{typeset}(A)$ to be $\{T_A(a) \mid 0 \neq a \in A\}$; A is *homogeneous* if $\text{typeset}(A)$ is a singleton. If $\text{rank } A = 1$, then A is homogeneous. Call A *completely decomposable* if A is the direct sum of rank 1 groups. Pure subgroups of finite rank homogeneous completely decomposable groups are summands.

If τ is a type, then $A(\tau) = \{x \in A \mid T_A(x) \geq \tau\}$ is a pure fully invariant subgroup of A . If $A = B \oplus C$, then $A(\tau) = B(\tau) \oplus C(\tau)$ and $A/A(\tau)$ is isomorphic to $B/B(\tau) \oplus C/C(\tau)$. Two abelian groups A and B are *quasi-isomorphic* if there are isomorphic subgroups A' and B' of A and B ,

respectively, such that A/A' and B/B' are bounded. The group A is *strongly indecomposable* if A quasi-isomorphic to $B \oplus C$ implies that $B=0$ or $C=0$.

1. $\mathcal{R}_\Lambda$ -groups.

LEMMA 1.1 (BUTLER [2]). Let A be an $\mathcal{R}$ -group with $\text{typeset}(A)$ where $\tau_0 = \tau_1 \cap \cdots \cap \tau_n$.

(a) If $\text{typeset}(A)$ is well ordered, then A is completely decomposable.

(b) If A is indecomposable, then $A/(A(\tau_1) + \cdots + A(\tau_n))$ is a finite group. Moreover, if $\tau_0 = \tau_i \cap \tau_j$ for $1 \leq i \neq j \leq n$, then $A(\tau_i) \cap A(\tau_j) = 0$ whenever $i \neq j$.

PROOF OF THEOREM 1. In view of Lemma 1.1(a), we may assume that $\text{typeset}(A) = \{\tau_0, \tau_1, \tau_2\}$ where $\tau_0 = \tau_1 \cap \tau_2$, $\tau_0 \neq \tau_1$, and $\tau_0 \neq \tau_2$. If A is indecomposable, then $A/(A(\tau_1) \oplus A(\tau_2)) = C_1 \oplus \cdots \oplus C_k$, where each C_i is a cyclic group of order m_i and m_i divides m_{i+1} . Let $A = A(\tau_1) + A(\tau_2) + Zx_1 + \cdots + Zx_k$, where $x_i + A(\tau_1) + A(\tau_2)$ is a generator of C_i and $m_i x_i = a_{1i} + a_{2i}$ for some $a_{ji} \in A(\tau_j)$ (Zx is the subgroup of A generated by x).

(a) Assume A is indecomposable with $\text{rank} > 1$ and let G_i be the pure subgroup of A generated by $\{a_{1i}, a_{2i}\}$ for $1 \leq i \leq k$. Note that a_{1i} and a_{2i} are both nonzero since $A(\tau_j)$ is a pure subgroup of A and $C_i \neq 0$. Thus $\text{rank}(G_i) = 2$.

Let $S = \{a_{11}, \cdots, a_{1k}, a_{21}, \cdots, a_{2k}\}$ and suppose that $l_1 a_{11} + \cdots + l_k a_{1k} + n_1 a_{21} + \cdots + n_k a_{2k} = 0$, for some $l_i, n_i \in \mathbb{Z}$. Since $A(\tau_1) \cap A(\tau_2) = 0$, $\sum l_i a_{1i} = \sum n_i a_{2i} = 0$ and $\sum l_i m_i x_i \in A(\tau_2)$. Thus $\sum r_i x_i \in A(\tau_2)$, where $r_i = (l_i m_i)/d$ and d is the greatest common divisor of $\{l_1 m_1, \cdots, l_k m_k\}$. Consequently, $\sum (r_i x_i + A(\tau_1) + A(\tau_2)) = 0 \in C_1 \oplus \cdots \oplus C_k$ and m_1 divides all r_i , a contradiction unless $l_1 = l_2 = \cdots = l_k = 0$. Similarly, $n_i = 0$ for $1 \leq i \leq k$ and S is a $\mathbb{Z}$ -independent subset of A .

One can easily verify that $G = G_1 \oplus \cdots \oplus G_k \subseteq A$ and that $A(\tau_j) = (G \cap A(\tau_j)) \oplus A'_j$ for some A'_j , where $j = 1, 2$ ($A(\tau_j)$ is a homogeneous completely decomposable group of type τ_j). But $A(\tau_1) \oplus A(\tau_2) \subseteq G \oplus A'_1 \oplus A'_2 \subseteq A$ and $x_1, \cdots, x_k \in G$ so $A = G_1 \oplus \cdots \oplus G_k \oplus A'_1 \oplus A'_2$. Since A is indecomposable, $A = G_k$, $k = 1$, and $A/(A(\tau_1) \oplus A(\tau_2))$ is cyclic.

(b) Assume that A is indecomposable and that $\tau_1 + \tau_2 = \tau_\infty$. If $qa \in A(\tau_1) \oplus A(\tau_2)$ for some prime q and some $a \in A$, then $a \in A(\tau_1) \oplus A(\tau_2)$ since either $qA(\tau_1) = A(\tau_1)$ or $qA(\tau_2) = A(\tau_2)$. Consequently, $A = A(\tau_1) \oplus A(\tau_2)$, a completely decomposable group.

(c) First assume that A is indecomposable of rank 2, and that $\tau_1 + \tau_2 = \tau_p$ for some prime p . Then $N_A = A/(A(\tau_1) \oplus A(\tau_2))$ is a cyclic group of order p^α for some $1 \leq \alpha < \infty$ (see Theorem 1(a) and 1(b)). Let B be another rank 2 indecomposable $\mathcal{R}_\Lambda$ -group such that $\text{typeset}(B) = \text{typeset}(A)$ where N_B is cyclic of order p^β .

If $\alpha = \beta$ then there is a monomorphism $f: A \rightarrow B$ such that $f: A(\tau_1) \oplus A(\tau_2) \rightarrow B(\tau_1) \oplus B(\tau_2)$ is an isomorphism. Let $A = A(\tau_1) + A(\tau_2) + Zx$, where $x + A(\tau_1) + A(\tau_2)$ is a generator of N_A , and let i be the least positive integer with $p^i f(x) \in B(\tau_1) \oplus B(\tau_2) = f(A(\tau_1) \oplus A(\tau_2))$. Since f is monic, $i = \alpha$ and f is an isomorphism. Thus $A \simeq B$ iff $\alpha = \beta$.

Let $A = A_1 \oplus \cdots \oplus A_m = B_1 \oplus \cdots \oplus B_n$ be two decompositions of A into indecomposable summands. Observe that N_A is the direct sum of a finite number of copies of cyclic p -groups and a finite number of rank 1 groups of type τ_0 . It now follows that $m = n$ and $A_i \simeq B_{\sigma(i)}$ for some permutation σ of $\{1, 2, \dots, n\}$ (use the preceding remarks and the fact that $N_A \simeq N_{A_1} \oplus \cdots \oplus N_{A_m} \simeq N_{B_1} \oplus \cdots \oplus N_{B_n}$).

PROOF OF THEOREM 2(a). Assume that A is indecomposable of rank ≥ 2 , denote $A(\tau_j)$ by A_j , $j = 1, 2, 3$, and write $A/(A_1 + A_2 + A_3) = C_1 \oplus \cdots \oplus C_k$, where C_i is a cyclic group of order m_i and m_i divides m_{i+1} . Then $A = A_1 + A_2 + A_3 + Zx_1 + \cdots + Zx_k$, where $x_i + A_1 + A_2 + A_3$ is a generator of C_i , $m_i x_i = x_{1i} + x_{2i} + x_{3i}$, and $x_{ji} \in A_j$.

Define $S_j = \{x_{ji} \neq 0 \mid 1 \leq i \leq k\}$ and let H_j be the pure subgroup of A generated by S_j , where $1 \leq j \leq 3$. We prove that $X = \{x_1, \dots, x_k\}$ may be chosen such that $H_j \cap A_{kl}^* = 0$, where $\{j, k, l\} = \{1, 2, 3\}$ and A_{kl}^* is the pure subgroup of A generated by $A_k + A_l$. Assume that, for example, $l_1 x_{11} + \cdots + l_k x_{1k} \in A_{23}^*$ for $l_i \in \mathbb{Z}$ and that there is a least integer t with $l_t x_{1t} \neq 0$. Then $\sum m_i l_i x_i \in A_{23}^*$ and $\sum r_i x_i \in A_{23}^*$ where, $r_i = (m_i l_i)/d$ and d is the greatest common divisor of $\{m_1 l_1, \dots, m_k l_k\}$. But $\sum A = \tau_\infty$ so there is an integer n with $n(\sum r_i x_i) \in A_2 + A_3$ and $nA_1 = A_1$. Since $x_i + A_1 + A_2 + A_3$ generates C_i , m_i divides nr_i for $1 \leq i \leq k$. But $r_i \neq 0$, consequently m_i divides n , and $m_i A_1 = A_1$. Define $x'_i = m_i^{-1}(x_{2i} + x_{3i})$ a generator of C_i . Replacing x_i by x'_i and repeating the argument for all nonzero $l_i x_{1i}$ proves that X may be chosen such that $H_1 \cap A_{23}^* = 0$. Since replacing x_i by x'_i does not affect S_2 and S_3 , similar arguments for S_2 and S_3 give the desired result. Note that, in fact, S_i is a $\mathbb{Z}$ -independent subset of A for $i = 1, 2, 3$.

Let G_i be the pure subgroup of A generated by $\{x_{1i}, x_{2i}, x_{3i}\}$. One easily proves that $G = G_1 \oplus \cdots \oplus G_k \subseteq A$. Now $A_3 = A'_3 \oplus (G \cap A_3)$ and $A_3 \subseteq G \oplus A'_3 \subseteq A$. Furthermore, $A_2 = (G \cap A_2) \oplus (A'_3 \cap A_2) \oplus A'_2$ and $A_2 + A_3 \subseteq G \oplus A'_3 \oplus A'_2$. Finally, $A_1 = (G \cap A_1) \oplus (A'_3 \cap A_1) \oplus (A'_2 \cap A_1) \oplus A'_1$ and $A_1 + A_2 + A_3 \subseteq G \oplus A'_3 \oplus A'_2 \oplus A'_1$. But $x_1, \dots, x_k \in G$, so $A = G_1 \oplus \cdots \oplus G_k \oplus A'_3 \oplus A'_2 \oplus A'_1$. Since A is indecomposable, $A = G_k$, $k = 1$, and $\text{rank } A \leq 3$.

COROLLARY 1.2. Let X_i be a rank 1 subgroup of Q such that $1 \in X_i$ and $\text{type}(X_i) = \tau_i$ for $i = 0, 1, 2, 3$ and let $V = Qx \oplus Qy \oplus Qz$ be a Q -vector space of dimension 3. If A is an indecomposable $\mathcal{R}_\Lambda$ -group and if $\sum \Lambda = \tau_\infty$, then A is isomorphic to a group in one of the following classes of subgroups of V :

- (i) X_0x, X_1x, X_2x, X_3x ;
- (ii) $X_1x + X_2y + X_0((x+y)/n_3)$ where $2 \leq n_3 < \infty$ and $n_3X_3 = X_3$;
- (iii) $X_1x + X_3y + X_0((x+y)/n_2)$ where $2 \leq n_2 < \infty$ and $n_2X_2 = X_2$;
- (iv) $X_2x + X_3y + X_0((x+y)/n_1)$ where $2 \leq n_1 < \infty$ and $n_1X_1 = X_1$;
- (v) $X_1x + X_2y + X_3(x+y)$;
- (vi) $X_1x + X_2y + X_3z + X_0((x+y+z)/n_1n_2n_3)$ where $2 \leq n_1n_2n_3 < \infty$ and $n_iX_i = X_i$.

LEMMA 1.3. Let A be an indecomposable $\mathcal{R}_\Lambda$ -group with $\text{typeset}(A) = \Lambda$ and assume that if p is a prime dividing the order of $A/(A(\tau_1) + A(\tau_2) + A(\tau_3))$, then $pA(\tau_j) = A(\tau_j)$ for some j . Then $A = A_{12}^* + A_{13}^* + A_{23}^*$ where A_{ij}^* is the pure subgroup of A generated by $A(\tau_i) + A(\tau_j)$.

PROOF. Let n be the smallest integer with $nA \subset A(\tau_1) + A(\tau_2) + A(\tau_3)$. The hypotheses guarantee the existence of relatively prime integers n_1, n_2, n_3 such that $n = n_1n_2n_3$ and $n_jA(\tau_j) = A(\tau_j)$.

Let $x = n^{-1}(a_1 + a_2 + a_3)$ be an arbitrary element of A for some $a_j \in A(\tau_j)$. Choose integers r_i and s_i with $r_1n_1 + s_1n_2 = r_2n_1 + s_2n_3 = r_3n_2 + s_3n_3 = 1$. Then $x = u + v + w$, where

$$u = n_3^{-1}(n_1^{-1}r_3a_1 + n_2^{-1}r_2a_2) \in A_{12}^*,$$

$$v = n_2^{-1}(n_1^{-1}s_3a_1 + n_3^{-1}r_1a_3) \in A_{13}^*,$$

$$w = n_1^{-1}(n_2^{-1}s_2a_2 + n_3^{-1}s_1a_3) \in A_{23}^*.$$

PROPOSITION 1.4. Assume that A is an indecomposable $\mathcal{R}_\Lambda$ -group with $\text{typeset}(A) = \Lambda$; that $A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3) \subset A$; and that if p is a prime dividing the order of $A/(A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3))$, then $pA(\tau_j) = A(\tau_j)$ for some j . Then $A \oplus A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3) \simeq A/A(\tau_1) \oplus A/A(\tau_2) \oplus A/A(\tau_3)$.

PROOF. Let $A_j = A(\tau_j)$. It suffices to construct a split exact sequence

$$(*) \quad 0 \longrightarrow A \xrightarrow{f_0} A/A_1 \oplus A/A_2 \oplus A/A_3 \xrightarrow{f_1} A/A_{12}^* \oplus A/A_{13}^* \oplus A/A_{23}^* \longrightarrow 0.$$

Note that A/A_{ij}^* is quasi-isomorphic, hence isomorphic, to A_k where $\{i, j, k\} = \{1, 2, 3\}$ (recall that A_k is homogeneous completely decomposable of type τ_k).

Define f_0 by letting $f_0(a) = (a + A_1, -a + A_2, a + A_3)$ and define

$$\begin{aligned} f_1(a + A_1, b + A_2, c + A_3) \\ = (a + b + A_{12}^*, c - a + A_{13}^*, -b - c + A_{23}^*). \end{aligned}$$

Clearly f_0 and f_1 are well defined, f_0 is monic and $\text{image } f_0 \subseteq \text{kernel } f_1$.

The image of f_0 is pure, for let p be a prime and let $pw=f_0(a)$ where $w=(x+A_1, y+A_2, z+A_3)$. Then $px-a=a_1 \in A_1$, $py+a=a_2 \in A_2$, and $pz-a=a_3 \in A_3$. Now $p(x+2y+z)=a_1+2a_2+a_3 \in A_1 \oplus A_2 \oplus A_3$. If $x+2y+z \in A_1 \oplus A_2 \oplus A_3$, then $p^{-1}a_j \in A(\tau_j)$, $p^{-1}a \in A$ and $w=f_0(p^{-1}a)$. If

$$x + 2y + z \notin A_1 \oplus A_2 \oplus A_3,$$

then $p^{-1}a_j \in A(\tau_j)$, for some j (recall hypotheses). Once again, $p^{-1}a \in A$ and $w=f_0(p^{-1}a)$.

The preceding remarks show that if kernel f_1 /image f_0 is torsion, then kernel f_1 =image f_0 . Let $w=(a+A_1, b+A_2, c+A_3) \in \text{kernel } f_1$. For some nonzero integer m , $m(a+b)=a_1+a_2 \in A_1 \oplus A_2$, $m(c-a)=b_1+b_3 \in A_1 \oplus A_3$ and $m(-b-c)=c_2+c_3 \in A_2 \oplus A_3$. But $0=m(a+b)+m(c-a)+m(-b-c)=(a_1+b_1)+(a_2+c_2)+(b_3+c_3) \in A_1 \oplus A_2 \oplus A_3$, so $a_1=-b_1$, $a_2=-c_2$ and $b_3=-c_3$. Consequently, $ma-a_1=-mb+a_2=mc-b_3$ and $f_0(ma-a_1)=(ma+A_1, mb+A_2, mc+A_3)=mw$. Thus kernel f_1 /image f_0 is torsion, as desired.

To prove that f_1 is epic, let $a+A_{12}^* \in A/A_{12}^*$ and write $a=a_{12}^*+a_{13}^*+a_{23}^*$ where $a_{ij}^* \in A_{ij}^*$. Then

$$\begin{aligned} f_1(a_{13}^* + A_1, a_{23}^* + A_2, 0) &= (a_{13}^* + a_{23}^* + A_{12}^*, -a_{13}^* + A_{13}^*, -a_{23}^* + A_{23}^*) \\ &= a + A_{12}^*. \end{aligned}$$

Similarly, A/A_{13}^* and A/A_{23}^* are contained in the image of f_1 .

Finally, we define a homomorphism $h: A/A_{12}^* \oplus A/A_{13}^* \oplus A/A_{23}^* \rightarrow A/A_1 \oplus A/A_2 \oplus A/A_3$. Note that $f_1: A_{13}^*/A_1 \oplus A_{23}^*/A_2 \rightarrow A/A_{12}^*$ is an epimorphism between two homogeneous completely decomposable groups of type τ_3 . Thus there is a homomorphism $h_3: A/A_{12}^* \rightarrow A_{13}^*/A_1 \oplus A_{23}^*/A_2$ with $f_1 h_3 = 1$ (see Fuchs [4, p. 163] or [1]). Similarly there are maps $h_2: A/A_{13}^* \rightarrow A_{12}^*/A_1 \oplus A_{23}^*/A_3$ and $h_1: A/A_{23}^* \rightarrow A_{13}^*/A_3 \oplus A_{12}^*/A_2$ with $f_1 h_2 = 1$ and $f_1 h_1 = 1$. Define $h = h_3 + h_2 + h_1$, and observe that $f_1 h = 1$. Thus (*) is a split exact sequence.

PROOF OF THEOREM 2(b). A consequence of the fact that $\sum_{\Lambda} = \tau_{\infty}$ and Proposition 1.4.

PROOF OF COROLLARY 3. (a) If A is a $\mathcal{C}_3$ -group, then $\text{typeset}(A) \subseteq \{\tau_p \cap \tau_q \cap \tau_r, \tau_p \cap \tau_q, \tau_q \cap \tau_r, \tau_p \cap \tau_r, \tau_p, \tau_q, \tau_r\}$. Suppose, for example, that $\tau_p \in \text{typeset}(A)$. Let R be a rank 1 subring of Q such that R has type τ_p (e.g., $R=Z_p$, the localization of Z at p). Then $A(\tau_p)$ is a free R -module, $R \otimes_Z A(\tau_p) \simeq A(\tau_p)$ and $R \otimes_Z A = F \oplus D$, where F is a free R -module and D is divisible. It follows that $R \otimes_Z A(\tau_p)$ is a summand of $R \otimes_Z A$. Under the canonical imbedding, $R \otimes_Z A(\tau_p) = A(\tau_p) \subset A \subset R \otimes_Z A$ so that $A(\tau_p)$ is a summand of A . Thus $A = A(\tau_p) \oplus A(\tau_q) \oplus A(\tau_r) \oplus G$ and $\tau_p, \tau_q, \tau_r \notin \text{typeset } G$.

(b) Let $\tau_1 = \tau_p \cap \tau_q$, $\tau_2 = \tau_p \cap \tau_r$, $\tau_3 = \tau_q \cap \tau_r$. If B is an indecomposable rank 2 $\mathcal{C}_3$ -group, then B is in class (see Corollary 1.2)

(ii) iff $B/(B(\tau_1) \oplus B(\tau_2)) \simeq Z/p^\alpha Z$, $1 \leq \alpha < \infty$;

(iii) iff $B/(B(\tau_1) \oplus B(\tau_3)) \simeq Z/q^\beta Z$, $1 \leq \beta < \infty$;

(iv) iff $B/(B(\tau_2) \oplus B(\tau_3)) \simeq Z/r^\gamma Z$, $1 \leq \gamma < \infty$; and

(v) iff $B/(B(\tau_1) \oplus B(\tau_2)) \simeq Z(p^\infty)$ (the p -divisible subgroup of Q/Z).

Let $A = G_1 \oplus \cdots \oplus G_k = H_1 \oplus \cdots \oplus H_l$ be a decomposition of A into indecomposable summands, where $\text{rank } G_i \leq 2$ and $\text{rank } H_i \leq 2$. Now $A/(A(\tau_1) + A(\tau_2))$ is a direct sum of a finite number of cyclic p -groups, a finite number of rank 1 groups of types τ_p , τ_q , τ_r , τ_0 and τ_3 and a finite number of copies of $Z(p^\infty)$. Moreover, $A/(A(\tau_1) + A(\tau_2) + A(\tau_3))$ is the direct sum of a finite number of cyclic p -groups, a finite number of cyclic q -groups, a finite number of cyclic r -groups, and a finite number of rank 1 groups of type τ_p , τ_q , τ_r and τ_0 . Thus $k = l$ and $G_i \simeq H_{\sigma(i)}$ for some permutation σ of $\{1, 2, \dots, k\}$.

(c) A consequence of Theorem 2(b) and Corollary 3(b).

2. Examples. Let $\Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ where $\tau_0 = \tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3$; let $\lambda = \{\tau_0, \tau_1, \tau_2\}$; let X_i be a rank 1 subgroup of Q with $1 \in X_i$ and with type $(X_i) = \tau_i$; and let $V = Qw \oplus Qx \oplus Qy \oplus Qz$.

EXAMPLE 2.1. If $\sum \lambda \neq \infty$, then there is a rank 2 indecomposable $\mathcal{R}$ -group A with typeset $(A) = \lambda$.

PROOF. Choose a prime p with $pX_i \neq X_i$, where $i = 1, 2$, and let $A = X_1w + X_2x + X_0((w+x)/p) \subset V$.

EXAMPLE 2.2. If $\sum \lambda \neq \tau_\infty$ and $\sum \lambda \neq \tau_p$ for all primes p , then there is an $\mathcal{R}$ -group A with typeset $(A) = \lambda$ such that $A = A_1 \oplus A_2 = A_3 \oplus B_1 \oplus B_2$, where each A_i is an indecomposable group of rank 2 and $B_1 \oplus B_2$ is completely decomposable.

PROOF. Choose distinct primes p and q with $pX_i \neq X_i$ and $qX_i \neq X_i$ for $i = 1, 2$. Define $A_1 = X_1w + X_2x + X_0((w+x)/p)$, $A_2 = X_1y + X_2z + X_0((y+z)/q)$ and let $A = A_1 \oplus A_2$. Let C_1 and C_2 be the pure subgroups of A generated by $qw + py$ and $qx + pz$, respectively. Define A_3 to be the pure subgroup of A generated by $C_1 \oplus C_2$ and note that $A_3(\tau_1) = C_1 \oplus B_1$ and $A_3(\tau_2) = C_2 \oplus B_2$, for some $B_i \subset A(\tau_i)$. Clearly $A_3 \cap (B_1 \oplus B_2) = 0$ so that $A(\tau_1) \oplus A(\tau_2) \subseteq A_3 \oplus B_1 \oplus B_2 \subseteq A$. But

$$a = p^{-1}(w+x) + q^{-1}(y+z) = ((qw+py) + (qx+pz))p^{-1}q^{-1} \in A_3$$

and the order of $a + A(\tau_1) + A(\tau_2)$ is pq in $A/(A(\tau_1) \oplus A(\tau_2))$ so that $A = A_1 \oplus A_2 = A_3 \oplus B_1 \oplus B_2$.

EXAMPLE 2.3. If $\sum \Lambda \neq \tau_\infty$, then there is an indecomposable rank 3 $\mathcal{R}$ -group A such that typeset $(A) = \Lambda$ and $A \oplus H$ is not a direct sum of

groups of rank ≤ 2 for all finite rank completely decomposable $\mathcal{H}_\Lambda$ -groups H .

PROOF. Choose a prime p with $pX_i \neq X_i$ for $i=1, 2, 3$ and let $A = X_1w + X_2x + X_3y + X_0((w+x+y)/p)$. Note that A is indecomposable and $(w+x+y)/p \notin A_{12}^* + A_{13}^* + A_{23}^*$. If $A \oplus H = G_1 \oplus \cdots \oplus G_k$, where H is a finite rank completely decomposable $\mathcal{H}_\Lambda$ -group and $\text{rank } G_i \leq 2$, then one can easily prove that $A = A_{12}^* + A_{13}^* + A_{23}^*$, a contradiction.

EXAMPLE 2.4. If $\sum_\Lambda \neq \tau_\infty$, then there is an indecomposable rank 3 $\mathcal{H}$ -group A such that $\text{typeset}(A) = \Lambda$ and A is not quasi-isomorphic to a completely decomposable group (see Corollary 1.2).

PROOF. Let $A' = X_1w + X_2x + X_3(w+x)$, a strongly indecomposable $\mathcal{H}_\Lambda$ -group of rank 2. Let $A = A' + X_3y + X_0((w+x+y)/p)$ where p is a prime with $pX_i \neq X_i$ for $i=1, 2, 3$. Then A is indecomposable of rank 3, A is quasi-isomorphic to $A' \oplus X_3$, and A' is not quasi-isomorphic to a completely decomposable group.

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