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MEASURABILITY OF LATTICE OPERATIONS IN A CONE

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ABSTRACT. Let X be a locally convex Hausdorff topological vector space and C a convex cone generating X such that C is a lattice in its own order. Under suitable conditions $(x, y)\rightarrow\sup(x, y)$ and $\inf(x, y)$ are shown to be measurable mappings.

Let X be a locally convex Hausdorff topological vector space over the real numbers. Let C be a closed proper convex cone with vertex 0 and let C generate X. Further, let C be a lattice in its own order. There are well-known results asserting the continuity of the mappings $(x, y)\rightarrow\sup(x, y)$ and $(x, y)\rightarrow\inf(x, y)$ under suitable restrictions on the cone C ([**4**, Chapter V], [**2**, Appendix]). In this note we shall give conditions under which the lattice operations are measurable mappings. This measurability was found to be very useful in our recent work in potential theory [**1**].

THEOREM 1. *Let X be a Hausdorff locally convex real topological vector space. Let C be a closed proper convex cone with vertex at the origin, generating X and such that C is a lattice in its own order. Let B be a compact metrizable base for C.*

Then, the mappings: $C\times C\rightarrow C$ given by $(x, y)\rightarrow\sup(x, y)$ and $(x, y)\rightarrow\inf(x, y)$ are Borel, viz., the inverse image of any Borel set of C under each of these mappings is a Borel set of $C\times C$.

PROOF. *Step* (1). Let us denote by K the set of all positive continuous linear functionals on X, and $Y=K-K$ the vector space generated by this cone. We note that Y separates the points of X [**4**, Example 25, p. 71] and hence $\sigma(x, y)$ on X is a Hausdorff topology. Let us denote by σ the topology induced on C by $\sigma(x, y)$ on X. Let τ be the given topology on X. We note that (C, τ) is locally compact, metrizable and separable and hence it is a polish space. It follows that the (C, σ) Borel sets and the (C, τ) Borel sets are identical and the same is the case on the product space $C\times C$ with the respective product topologies [**5**].

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Step (2). Let L_0 be the positive continuous linear functional on X defining the base B, i.e., $B=\{x \in C: L_0(x)=1\}$ [**3**, Theorem 3.6]. Let x and y be any two elements of C and $L_0(x+y)=a$. The set of all elements z in C majorizing both x and y and such that $z \leqq x+y$ is decreasingly directed and is contained in the τ-compact set $\{z \in C: L_0(z) \leqq a\}$. Therefore, it is possible to choose a decreasing sequence $\{z_n\}$ satisfying (*i*) $z_n \leqq x+y$ for every n, and (ii) $\{z_n\}$ τ-converges to $\sup(x, y)$.

Step (3). Let us fix a positive continuous linear functional L on X. We shall show that $(x, y) \rightarrow L[\sup(x, y)]$ (resp. $(x, y) \rightarrow L[\inf(x, y)]$) is a lower (resp. upper) semicontinuous function on $C \times C$. It is sufficient to prove the lower semicontinuity of the first mapping. For this, consider (x_n, y_n) in $C \times C$ converging to (x_0, y_0) such that x_n, y_n for all n and x_0, y_0 belong to a fixed compact set A of C, A of the form $\{z: L_0(z) \leqq b\}$. Given $\varepsilon > 0$, it is possible to choose an element s_n, for every n, such that (i) s_n belongs to $A+A$, (ii) s_n majorizes both x_n and y_n and (iii) $L[\sup(x_n, y_n)] > L(s_n) - \varepsilon/2$. Hence, given any subsequence, $\{n \in M' \subset N\}$, we may choose a further subsequence, say $\{n \in M \subset M' \subset N\}$, such that as n in M tends to infinity, $\{s_n\}$ converges to an element z in C. Hence, for all sufficiently large n in M, $L(s_n) > L(z) - \varepsilon/2$ and consequently, $L[\sup(x_n, y_n)] > L(z) - \varepsilon$.

We now claim that z is a majorant of $\sup(x_0, y_0)$. Since s_n is a majorant of both x_n and y_n, there are elements t_n and u_n in C such that $s_n = x_n + t_n$ and $s_n = y_n + u_n$ for every n. It is clear that $\{t_n\}$ and $\{u_n\}$ converge respectively to t_0 and u_0 in C as n in M tends to infinity. Hence, $z = x_0 + t_0$ and $z = y_0 + u_0$ proving the assertion $z \geqq \sup(x_0, y_0)$. We deduce from the above inequalities that for all sufficiently large n in M,

$$L[\sup(x_n, y_n)] > L(z) - \varepsilon \geqq L[\sup(x_0, y_0)] - \varepsilon.$$

This proves the lower semicontinuity of the function $(x, y) \rightarrow L[\sup(x, y)]$.

Step (4). Define for any L in K and nonnegative real numbers a and b, $W(L, a, b) = \{x \in C: a < L(x) < b\}$. The sets of the form $W(L, a, b)$ constitute a subbasis for the open sets of (C, σ). However, (C, σ) is a strongly Lindelöf space, i.e., every open subspace is Lindelöf. Hence, each τ' open set in C is a countable union of finite intersections of sets of the form $W(L, a, b)$. Hence, the sets of the form $W(L, a, b)$ generate the Borel σ-algebra of (C, σ). But

$$\{(x, y) \in C \times C: \sup(x, y) \in W(L, a, b)\} = \{(x, y) \in C \times C: a < L[\sup(x, y)] < b\}$$

is a Borel subset of $C \times C$, proving the Borel measurability of $(x, y) \rightarrow \sup(x, y): C \times C \rightarrow C$. The Borel measurability of the infimum is deduced in a similar way from the upper semicontinuity of $(x, y) \to L[\inf(x, y)]$. The theorem is proved.

EXAMPLE. Let H^+ be the set of all positive harmonic functions on the unit disc U and let H be the differences of elements in H^+. Then, H provided with the topology of uniform convergence on compact subsets of U is a locally convex space and it is well known that H and H^+ satisfy all the conditions of the above theorem. We shall show that the lattice operations are not continuous on H^+. Let $P = e^{i\theta}$ and $P_n = e^{i\theta_n}$ be points on the unit circle such that P_n converges to P and $P_n \neq P_m$ if $n \neq m$. Let $u_n(r, \varnothing)$ be the Poisson function with pole at θ_n, i.e.,

$$u_n(r, \varnothing) = (1 - r^2)/[1 - 2r\cos(\theta_n - \varnothing) + r^2].$$

Then u_n belongs to H^+, and it is clear that u_n converges to u, the Poisson function with pole at $e^{i\theta}$. Since u_n is an extreme element of the base $B = \{w \in H^+ : w(0) = 1\}$ (and so is u), it is easy to see that $\sup(u_n, u) = u_n + u$ and hence $\sup(u_n, u)$ converges to $2u$ which is not equal to $u = \sup(\lim u_n, u)$.

It is natural to raise the question whether the various mappings $x \to x^+$, $x \to x^-$, $(x, y) \to \sup(x, y)$ and $(x, y) \to \inf(x, y)$ are measurable. It is easily verified that the measurability of one of them is equivalent to the measurability of all the mappings. Now, we have the following result.

THEOREM 2. *Let X and C be as in the earlier theorem. Then the mapping $x \to x^+$ is μ-Lusin measurable for every locally finite Borel measure μ on X.*

PROOF. We observe that X is a Suslin space, since it is the image of the polish space $C \times C$ under the continuous mapping $(x, y) \to x - y$. Hence, there is a mapping $\varphi : X \to C \times C$ such that φ composed with $(x, y) \to x - y$ is identity on X and φ is μ-Lusin measurable for every Radon measure μ on X [5].

Let now K be a compact set of X and μ a locally finite Borel measure on X. Then μ is a Radon measure [5]. Given $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset K$ such that $\mu(K_\varepsilon) > \mu(K) - \varepsilon$ and φ restricted to K_ε is continuous. Hence, φ restricted to K_ε is a homeomorphism onto $\varphi(K_\varepsilon)$. From this we deduce that $\{|x| : x \in K_\varepsilon\}$ is contained in a compact set of C. Also, since K_ε is compact and Suslin it is a metrizable space. Now, exactly as in the proof of Theorem 1, it can be shown that $x \to x^+ : K_\varepsilon \to C$ is Borel measurable. But C being a polish space, we deduce that $x \to x^+$ is, in fact, a μ-Lusin measurable function on X. The proof is complete.

REMARK. For X and C as above, it can be shown that there is always a point of discontinuity of the mapping $x \to x^+$.

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