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n-GORENSTEIN RINGS

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ABSTRACT. The object of this note is to study commutative noetherian n-Gorenstein rings. The first result is: if each module satisfying Samuel's conditions (a_i) for some $i \leqq n$ is an ith syzygy, then the ring is n-Gorenstein. This is the converse to a theorem of Ischebeck. The next result characterizes n-Gorenstein rings in terms of commutativity of certain rings of endomorphisms. This answers a question of Vasconcelos. Finally the last result deals with embedding of finitely generated modules into finitely generated modules of finite projective dimension.

1. **Notation.** Throughout this note A will denote a commutative noetherian ring. Modules will in general be finitely generated (=f.g.)—the exceptions will be easy to identify. For a module M and a prime ideal $\mathfrak{p}$ let depth $M_\mathfrak{p}$ denote the $A_\mathfrak{p}$-depth ($=A_\mathfrak{p}$-homological codimension) of $M_\mathfrak{p}$, and let $\operatorname{gr}(\mathfrak{p})$ denote the grade of $\mathfrak{p}$ (=maximal length of A-regular sequence in $\mathfrak{p}$). Recall that $\operatorname{gr}(\mathfrak{p})=\inf\{\operatorname{depth} A_\mathfrak{q} \mid \mathfrak{q} \in V(\mathfrak{p})\}$.

For a module M we define $\Omega^i(M)$, $i \geqq 0$, and $D(M)$ as follows:

$$\Omega^i(M) = \operatorname{Coker}(P_{i+1} \to P_i)$$
$$D(M) = \operatorname{Coker}(P_0^* \to P_1^*)$$

where

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

is any projective resolution of M with each P_i f.g. and $*=\operatorname{Hom}(-, A)$. Furthermore write $E^i=\operatorname{Ext}^i(-, A)$. $\Omega^i(M)$ and $D(M)$ depends of course on the choice of projective resolution, but $\Omega^i(M)$ is unique up to projective equivalence and $E^i(D(M))$ is unique (cf. [**1**, Chapter 2, §1]).

2. **Four properties on a module.** For a module M and an integer $n \geqq 0$ we will consider the following four properties on M:

(a_n) Each A-regular sequence of length at most n is also M-regular (for $n=1$ this means simply: M is torsion free).

(b_n) depth $M_\mathfrak{p} \geqq \inf(n, \operatorname{depth} A_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ ($n=1$: $\operatorname{Ass}(M) \subseteqq \operatorname{Ass}(A)$).

(s_n) M is an nth syzygy, that is: $M=\Omega^n(L)$ for a suitable L (and a suitable projective resolution) ($n=1$: M is torsionless).

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(t_n) M is without n-torsion (or n-torsion free), that is: $E^i(D(M))=0$ for $1 \leqq i \leqq n$ ($n=1$: M is torsionless; $n=2$: M is reflexive, [**1**, (2.1), p. 48]).

The next four results are known and included to show the relations between these four properties ($n \geqq 0$ is a fixed integer in what follows).

2.1. PROPOSITION. *For all modules* (*still f.g.*) *we have*:

$$(t_n) \Rightarrow (s_n) \Rightarrow (b_n) \Rightarrow (a_n).$$

Cf. [**1**, Theorem 2.17] *and* [**6**, Satz 4.4].

2.2. PROPOSITION. (a_n) *and* (b_n) *are equivalent for all modules if A satisfies Serre's condition*:

(S_n) depth $A_\mathfrak{p} \geqq \inf(n, \mathrm{ht}(\mathfrak{p}))$ *for all prime ideals* (*or equivalently*: $A_\mathfrak{p}$ *is a Cohen-Macaulay ring for all prime ideals* $\mathfrak{p}$ *with* depth $A_\mathfrak{p} < n$), *cf.* [**8**, *Proposition* 6].

In Proposition 3.3, we shall see that the equivalence of (a_n) and (b_n) characterizes a slightly wider class of rings than the (S_n)-rings.

2.3. PROPOSITION. *All four properties are equivalent* (*i.e.* (a_n)$\Rightarrow$(t_n)) *if A is n-Gorenstein, that is*:

(G_n) $A_\mathfrak{p}$ *is Gorenstein for all prime ideals* $\mathfrak{p}$ *with* depth $A_\mathfrak{p} < n$.

Cf. [**6**, 4.6 Satz]. It will be proved later that the converse of this also holds. This answers Question 4.8 of Ischebeck [**6**], but is already known in the following special cases:

If $n=1$ it is proved by Vasconcelos [**9**, Theorem A.1].

If $n=2$ and A satisfies (S_2), then it is proved by Fossum and Reiten [**3**, Proposition 9].

If A is Cohen-Macaulay and is a homomorphic image of a Gorenstein ring then it is proved for all n in [**4**, Proposition 3.2].

In [**3**] several equivalent conditions on n-Gorenstein rings are examined, we shall use the following (see [**1**, Proposition 4.21]):

2.4. PROPOSITION. *Each* ($n+1$)*th syzygy is without* ($n+1$)*-torsion if and only if A is n-Gorenstein.*

3. **The properties** (a_n) **and** (b_n). From the proof of [**8**, Proposition 6] one easily obtains:

3.1. LEMMA. *M is an* (a_n)*-module if and only if* depth $M_\mathfrak{p} \geqq \inf(n, \mathrm{gr}(\mathfrak{p}))$ *for all* $\mathfrak{p}$.

The next two lemmas give examples of (a_n)- and (b_n)-modules:

3.2a. LEMMA. *If all* $\mathfrak{p} \in \mathrm{Ass}(M)$ *have* $\mathrm{gr}(\mathfrak{p}) < n$ *then* $\Omega^{n-1}(M)$ *is an* (a_n)*-module.*

PROOF. Since $\Omega^{n-1}(M)$ is a $(n-1)$th *syzygy* it is a fortiori an (a_{n-1})-module (Proposition 2.1). Therefore assume $\mathrm{gr}(\mathfrak{p}) \geqq n$ (and we want to show depth $\Omega^{n-1}(M)_{\mathfrak{p}} \geqq n$). It follows from the homological characterization of depth that depth $\Omega^{n-1}(M)_{\mathfrak{p}} \geqq \inf(\text{depth } A_{\mathfrak{p}}, \text{ depth } M_{\mathfrak{p}}+n-1) \geqq n$, since depth $M_{\mathfrak{p}} \geqq 1$ by assumption.

3.2b. LEMMA. *If all* $\mathfrak{p} \in \mathrm{Ass}(M)$ *have* depth $A_{\mathfrak{p}} < n$ *then* $\Omega^{n-1}(M)$ *is a* (b_n)*-module.*

3.3. PROPOSITION. *The following statements are equivalent*:

(1) *For all* $i \leqq n$ *each* (a_i)*-module satisfies* (b_i).

(2) depth $A_{\mathfrak{p}} \geqq \inf(n, \text{ depth } A_{\mathfrak{q}})$ *for all prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *where* $\mathfrak{p} \supseteqq \mathfrak{q}$.

(3) depth $A_{\mathfrak{p}} \geqq \inf(n, \mathrm{ht}(\mathfrak{p})-1)$ *for all prime ideals* $\mathfrak{p}$.

(4) $\mathrm{gr}(\mathfrak{p}) \geqq \inf(n, \text{ depth } A_{\mathfrak{p}})$ *for all prime ideals* $\mathfrak{p}$.

(5) $\mathfrak{p} \in \mathrm{Ass}(A/(a_1, \cdots, a_g))$ *for all maximal A-regular sequences* $a_1, \cdots, a_g$ *in a prime ideal* $\mathfrak{p}$ *with* $g = \mathrm{gr}(\mathfrak{p}) < n$.

(6) *Each prime ideal of grade less than n is an associated prime of a module of finite G-dimension* (*for definition see* [1, *Chapter* 3]).

PROOF. (1) implies (2). Assume $\mathfrak{q} \subseteqq \mathfrak{p}$, $d = \text{depth } A_{\mathfrak{p}} < n$. $\Omega^d(A/\mathfrak{q})$ is by Lemma 3.2a an (a_{d+1})-module and hence also a (b_{d+1})-module by assumption. Now depth $\Omega^d(A/\mathfrak{q})_{\mathfrak{q}} = \text{depth } \Omega^d_{A_{\mathfrak{q}}}(k(\mathfrak{q})) \leqq d < d+1$, and hence depth $A_{\mathfrak{q}} \leqq d$ as desired.

(2) *implies* (3). By [2, Corollary 5.3] we have

$$\mathrm{ht}(\mathfrak{p}) \leqq \sup\{\text{depth } A_{\mathfrak{q}} + 1 \mid \mathfrak{q} \subseteqq \mathfrak{p}\}.$$

(3) *implies* (4). Assume $g = \mathrm{gr}(\mathfrak{p}) = \text{depth } A_{\mathfrak{q}} < n$, $\mathfrak{q} \in V(\mathfrak{p})$. If $\mathfrak{p} = \mathfrak{q}$ we are done, and if $\mathfrak{p} \neq \mathfrak{q}$ we have depth $A_{\mathfrak{p}} \leqq \mathrm{ht}(\mathfrak{p}) \leqq \mathrm{ht}(\mathfrak{q}) - 1 \leqq \text{depth } A_{\mathfrak{q}} = g$.

(4) *implies* (5). Let $a_1, \cdots, a_g \in \mathfrak{p}$ be an A-regular sequence, $g = \mathrm{gr}(\mathfrak{p}) < n$. Then $g = \text{depth } A_{\mathfrak{p}} = \text{depth}(A_{\mathfrak{p}}/\boldsymbol{a}A_{\mathfrak{p}}) + g$, where $\boldsymbol{a} = (a_1, \cdots, a_g)$, and hence $\text{depth}(A/\boldsymbol{a})_{\mathfrak{p}} = 0$, i.e. $\mathfrak{p} \in \mathrm{Ass}(A/\boldsymbol{a})$.

(6) *implies* (4). Assume $g = \mathrm{gr}(\mathfrak{p}) = \text{depth } A_{\mathfrak{q}} < n$, $\mathfrak{q} \in V(\mathfrak{p})$. Then $g = \text{depth } A_{\mathfrak{q}} \geqq G\text{-dim } M_{\mathfrak{q}} \geqq G\text{-dim } M_{\mathfrak{p}} = \text{depth } A_{\mathfrak{p}} - \text{depth } M_{\mathfrak{p}} = \text{depth } A_{\mathfrak{p}}$, cf. [1, Chapter 3].

That (5) implies (6) and that (4) implies (1) is obvious (cf. respectively [1, Chapter 3] and Lemma 3.1).

4. Equivalence of all four properties.

4.1. THEOREM. *The following statements are equivalent*:

(1) *A is an n-Gorenstein ring.*

(2) *Each* (a_n)*-module is without n-torsion.*

(3) *For all i*, $1 \leqq i \leqq n$, *each* (a_i)*-module is an ith syzygy.*

(4) *Each* (b_n)*-module is without n-torsion.*

(5) *For all i*, $1 \leqq i \leqq n$, *each* (b_i)*-module is an ith syzygy.*

PROOF. That (1) implies both (2) and (3) follows from Proposition 2.3, and from Proposition 2.1 we know that (4) and (5) are weaker than respectively (2) and (3).

(4) *implies* (5). Since each nth syzygy is a (b_n)-module and hence without n-torsion by assumption, we conclude that A is $(n-1)$-Gorenstein, by Proposition 2.4. By Proposition 2.3 we get that for all $i<n$ each (b_i)-module is an ith syzygy, and a (b_n)-module is already an nth syzygy by assumption.

(5) *implies* (1). First we treat the case $n=1$. For $\mathfrak{p} \in \operatorname{Ass}(A)$ let $\mathfrak{p}^{(i)}$ denote the ith symbolic power of $\mathfrak{p}$. $A/\mathfrak{p}^{(i)}$ is a (b_1)-module and hence torsionless. Therefore $\mathfrak{p}^{(i)}=\operatorname{Ann} x_i$ for a suitable element x_i in some free A-module, and we conclude that $\mathfrak{p}^{(i)}=\operatorname{Ann}(\mathfrak{A}_i)$ for a suitable ideal $\mathfrak{A}_i$. This gives $\operatorname{Ann}(\operatorname{Ann}(\mathfrak{p}^{(i)}))=\mathfrak{p}^{(i)}$. From the descending chain: $\mathfrak{p}^{(1)}\supseteqq\mathfrak{p}^{(2)}\supseteqq\cdots\supseteqq\mathfrak{p}^{(i)}\supseteqq\cdots$ we obtain an ascending chain: $\operatorname{Ann}(\mathfrak{p}^{(1)})\subseteqq\operatorname{Ann}(\mathfrak{p}^{(2)})\subseteqq\cdots\subseteqq\operatorname{Ann}(\mathfrak{p}^{(i)})\subseteqq\cdots$. That is: $\operatorname{Ann}(\mathfrak{p}^{(i)})=\operatorname{Ann}(\mathfrak{p}^{(i+1)})$ for i big enough, and hence $\mathfrak{p}^{(i)}=\mathfrak{p}^{(i+1)}$. By passing to the local ring $A_\mathfrak{p}$ we obtain: $(\mathfrak{p}A_\mathfrak{p})^i=\mathfrak{p}^{(i)}A_\mathfrak{p}=\mathfrak{p}^{(i+1)}A_\mathfrak{p}=(\mathfrak{p}A_\mathfrak{p})^{i+1}$, and hence $(\mathfrak{p}A_\mathfrak{p})^i=0$, i.e. $\operatorname{ht}(\mathfrak{p})=0$. Now it is proved that all associated primes are of height 0, that is: A is an (S_1)-ring, and hence $(\mathrm{a}_1)\Leftrightarrow(\mathrm{b}_1)$, so by Vasconcelos [**9**, Theorem A.1] A is 1-Gorenstein. This completes the proof in the case $n=1$

Now back to the proof in the general case: $n\geqq 1$. Let $a_1,\cdots,a_i$, $i<n$, be any A-regular sequence. For an A-module X we will write $\bar{X}=X/(a_1,\cdots,a_i)X$. We are going to prove that any (b_1)-$\bar{A}$-module is a torsionless $\bar{A}$-module, because this will imply that $\bar{A}$ is 1-Gorenstein, and hence that A is n-Gorenstein by [**6**, Theorem 3.15].

Let M be any (b_1)-$\bar{A}$-module. If $\mathfrak{p} \in \operatorname{Ass}_A(M)$ then

$$\mathfrak{p}' = \mathfrak{p}/(a_1,\cdots,a_i) \in \operatorname{Ass}_{\bar{A}}(\bar{M}) \subseteqq \operatorname{Ass}_{\bar{A}}(\bar{A}),$$

i.e. depth $A_\mathfrak{p}=i+\operatorname{depth}\bar{A}_{\mathfrak{p}'}=i<i+1$. By Lemma 3.2b $K=\Omega_A^i(M)$ is a (b_{i+1})-A-module and hence an $(i+1)$th *syzygy* by our assumption. Let $0\to K\to F\to C\to 0$ be exact with F projective and C an ith syzygy. We have $\operatorname{Tor}_1^A(C,\bar{A})=0$, and hence an exact sequence $0\to\bar{K}\to\bar{F}$. This shows that $\bar{K}$ is a torsionless $\bar{A}$-module. Furthermore we have $\operatorname{Tor}_1^A(\Omega_A^{i-1}(M),\bar{A})=\operatorname{Tor}_i^A(M,\bar{A})=M$, the last isomorphism holds because $a_1,\cdots,a_i \in \operatorname{Ann}_A(M)$. Now, since $K=\Omega_A^i(M)$ there is an exact sequence $0\to K\to F\to\Omega_A^{i-1}(M)\to 0$ with F free. Tensoring with $\bar{A}$ gives an exact sequence $0\to M\to\bar{K}$, that is: M is a submodule of a torsionless $\bar{A}$-module, and hence M is torsionless as an $\bar{A}$-module and this was what we desired.

5. **Commutative endomorphism rings.** In [**10**] Vasconcelos conjectured that the ring of endomorphisms $\operatorname{End}_A(\mathfrak{A})$ is commutative for all ideals $\mathfrak{A}$ if and only if A is 1-Gorenstein. This is proved in the following:

5.1. THEOREM. *The ring A is n-Gorenstein if and only if: For each ideal $\mathfrak{A}$ in A and each A-regular sequence $a_1, \cdots, a_i \in \mathfrak{A}$ with $0 \leqq i < n$ the ring of endomorphisms* $\mathrm{End}_A(\mathfrak{A}/(a_1, \cdots, a_i))$ *is commutative.*

PROOF. By [6, Theorem 3.15] it is enough to prove that A is 1-Gorenstein if and only if $\mathrm{End}(\mathfrak{A})$ is commutative for all $\mathfrak{A}$, and this will follow from:

5.2. LEMMA. *Let M be a nonzero A-module. Then the following are equivalent:*

(1) $\mathrm{End}_A(N)$ *is commutative for all submodules N of M.*

(2) $\mathrm{Ass}(M)$ *is without embedded primes and $M_\mathfrak{p}$ has simple socle over $A_\mathfrak{p}$ for all* $\mathfrak{p} \in \mathrm{Ass}(M)$.

PROOF. (1) *implies* (2). Let N_1 and N_2 be submodules of M such that $N_1 \cap N_2 = 0$. Since $\mathrm{End}(N_1 \oplus N_2)$ is commutative it is easy to see $\mathrm{Hom}(N_1, N_2) = 0$. This shows that for all distinct associated primes $\mathfrak{p}$ and $\mathfrak{q}$ (of M) we have $\mathrm{Hom}(A/\mathfrak{p}, A/\mathfrak{q}) = 0$ and hence $\mathfrak{p}$ cannot be contained in $\mathfrak{q}$. Let S be the (nonzero) socle of the (artinian) $A_\mathfrak{p}$-module $M_\mathfrak{p}$. Then $\mathrm{End}_{A_\mathfrak{p}}(S)$ is commutative and (as above) we conclude that S is indecomposable, i.e. S is simple.

(2) *implies* (1). Let $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$. Then $E(M) = E(A/\mathfrak{p}_1) \oplus \cdots \oplus E(A/\mathfrak{p}_r)$ by the last part of our assumption (2). $\mathrm{End}(E(A/\mathfrak{p}))$ is commutative, since it is ring-isomorphic to the $\mathfrak{p}A_\mathfrak{p}$-adic completion of $A_\mathfrak{p}$ (cf. [7, Theorem 3.7]). By the first part of our assumption

$$\mathrm{Hom}(E(A/\mathfrak{p}_i), E(A/\mathfrak{p}_j)) = 0 \quad \text{for } i \neq j.$$

It is now justified that $\mathrm{End}(E(M))$ is commutative. Now let N be any submodule of M. Since every endomorphism of N can be extended to an endomorphism of $E(M)$ we conclude that $\mathrm{End}(N)$ is commutative.

6. **Embedding of modules over n-Gorenstein rings.** After some modifications of the proof of Theorem 2 in [5] it is possible to prove:

6.1. THEOREM. *Let A be a homomorphic image of a Gorenstein ring. Then A is n-Gorenstein if and only if A is* (S_n) *and each f.g. module M such that* $\mathrm{gr}(\mathfrak{p}) < n$ *for all* $\mathfrak{p} \in \mathrm{Ass}(M)$ *is embeddable in a f.g. module of finite projective dimension.*

It would be nice to avoid the condition that A is (S_n); if the embeddability-condition holds A satisfies at least: depth $A_\mathfrak{p} \geqq \inf(n, \mathrm{ht}(\mathfrak{p}) - 1)$ by Proposition 3.3.

ADDED IN PROOF. Theorem 5.1 has already been proved in the main case $n=1$ by S. Alamelu, Proc. Amer. Math. Soc. **37** (1973), 29–31.

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