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ON TORSION ABELIAN GROUPS QUASI-PROJECTIVE OVER THEIR ENDOMORPHISM RINGS

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ABSTRACT. It is shown that a torsion abelian group is quasi-projective over its endomorphism ring exactly if, for every prime p, its p-component is bounded or has an unbounded basic subgroup.

Recently, a number of papers have dealt with the behavior of abelian groups A regarded as modules over their endomorphism rings $E(A)=E$. For instance, Richman and Walker [3] have shown that an abelian p-group is flat as an E-module if and only if it is either bounded or has an unbounded basic subgroup. In another paper [4], they described all abelian groups which are injective as modules over their endomorphism rings. Poole and Reid [1] raised the question of abelian groups quasi-injective over their endomorphism rings; they have shown that all unmixed divisible groups and direct sums of finite cyclic groups share this property. Subsequently, Richman [2] proved that the class of p-groups, quasi-injective over their respective endomorphism rings, is fairly large: it includes all p-groups without elements of infinite height, the totally projective p-groups and all nonreduced p-groups.

In this note, we wish to raise the dual question: Which abelian groups are quasi-projective over their endomorphism rings? This question can be fully answered for torsion groups by the following simple result.

THEOREM.[2] *A torsion abelian group is quasi-projective as a module over the ring of its endomorphisms if and only if, for every prime p, its p-component is either bounded or has an unbounded basic subgroup.*

It is a routine exercise to show that it suffices to consider p-groups A.

Suppose that A is a p-group as stated, G is an E-submodule of A (i.e. a fully invariant subgroup of A) and $\phi: A\to A/G$ is the natural homomorphism. What we have to prove is that for every E-homomorphism $\alpha: A\to A/G$,

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[2] This result immediately generalizes to torsion modules over Dedekind domains.

there exists an endomorphism λ of A making the diagram

$$\begin{tikzcd} & A \arrow[dl, dashed, "\lambda"'] \arrow[d, "\alpha"] \\ A \arrow[r, "\phi"] & A/G \end{tikzcd}$$

commutative. The hypothesis of α being an E-map is equivalent to saying that, for every $\eta \in E$,

$$\bar{\eta}\alpha = \alpha\eta \tag{1}$$

where $\bar{\eta}$ denotes the map $A/G \to A/G$ induced by η, i.e. $\bar{\eta}(a+G)=\eta a+G$ for $a \in A$.

Let $\langle a \rangle$ be a summand, of order p^n, of A, and write $A=\langle a \rangle \oplus C$. Then the full invariance of G implies $G=(\langle a \rangle \cap G)\oplus(C \cap G)$, and hence we get a direct decomposition

$$A/G = (\langle a \rangle + G)/G \oplus (C + G)/G.$$

Let η be the endomorphism of A which is multiplication by an integer t on C and which satisfies $\eta a=a+x$ where x is some element in $C[p^n]$. We can write

$$\alpha a = k\bar{a} + \bar{c}$$

where bars indicate cosets mod G, $c \in C$ and k is an integer. Since $\bar{\eta}\bar{a}= \bar{a}+\bar{x}$, $\bar{\eta}\bar{c}=t\bar{c}$, from (1) we deduce that

$$\bar{\eta}\alpha a = \bar{\eta}(k\bar{a} + \bar{c}) = k\bar{a} + k\bar{x} + t\bar{c},$$
$$\alpha\eta a = \alpha(a + x) = k\bar{a} + \bar{c} + \alpha x$$

are equal, for every t. Therefore $\bar{c}=0$, $\alpha a=k\bar{a}$ and

$$\alpha x = k\bar{x} \quad \text{for all } x \in A[p^n]. \tag{2}$$

Suppose that A has a summand $\langle b \rangle$ of order $>p^n$. Then we get similarly $\alpha x=k'\bar{x}$ for some integer k' and for all $x \in A[o(b)]$, thus $k\equiv k' \bmod o(\bar{a})$. It is now easy to conclude that if A has an unbounded basic subgroup, then there is a p-adic integer π such that $\alpha x=\pi\bar{x}$ for all $x \in A$. Consequently, if we choose λ in the diagram to be the multiplication by this π, then the arising triangle will commute.

If A is a bounded p-group and if $\langle a \rangle$ is a cyclic summand of maximal order, then (2) shows that λ can be chosen as a multiplication by k. (Bounded p-groups are actually projective as E-modules; see Richman and Walker [3].)

In the remaining case, A is of the form $A=B\oplus D$ where B is bounded (say, $p^{m-1}B=0$) and D is divisible $\neq 0$. Let $G=A[p^m]$ and select an E-homomorphism $\alpha: A \to A/G$ such that $\alpha B=0$ and $\alpha|D$ is monic. This can be done, for instance, by first writing $D=\bigoplus_{i\in I} D_i$ with $D_i=\langle c_{i1}, \cdots, c_{in}, \cdots\rangle\cong Z(p^\infty)$, $pc_{i1}=0$, $pc_{i,n+1}=c_{in}$ $(n\geqq 1)$, and then setting $\alpha c_{in}=\bar{c}_{i,n+m}$. For such an α, there is no $\lambda:A\to A$ with $\phi\lambda=\alpha$, since $\alpha D[p^m]\neq 0$, but $\phi\lambda D[p^m]$ must vanish. This completes the proof.

In view of [3], we conclude:

COROLLARY. *A torsion group is quasi-projective over its endomorphism ring E exactly if it is a flat E-module.*

The above method can be suitably generalized to describe the algebraically compact and cotorsion groups which are quasi-projective over their endomorphism rings (see a forthcoming paper by A. Longtin).

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