ON THE ABSOLUTE SUMMABILITY OF THE ALLIED INTEGRAL BY A FUNCTIONAL NÖRLUND METHOD

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ABSTRACT. In this paper a theorem on the absolute summability of an integral associated with the allied integral of a Fourier integral by a functional Nörlund method has been established. Our theorem generalises an earlier result due to Boicun on the absolute Nörlund summability of the allied integral.

1. Let $p(t)$ be a real locally integrable function on the real half-line. If f is locally integrable in $(0, \infty)$,

$$P(y) = \int_0^y p(t) dt,$$

and

$$\sigma(y) = \frac{1}{P(y)} \int_0^y P(y-u)f(u) du,$$

then the integral $\int_0^\infty f(u) du$ is said to be absolutely $(N, p(y))$ summable or summable $|N, p(y)|$ if $\int_0^\infty |\sigma'(y)| dy < C$.¹

With $p(t) = \alpha t^{\alpha-1}$ ($\alpha > 0$) the method of summability $|N, p(y)|$ is the same as the well-known Cesàro's method $|C, \alpha|$.

2. Let $f(t) \in L(-\infty, \infty)$. The allied integral of a Fourier integral of the function $f(t)$ is defined [3]

$$(2.1) \quad \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(t) \sin(t-x)u dt.$$

The absolute Nörlund summability of Fourier series and its allied series has been studied by several workers. The corresponding problem of absolute summability of trigonometric integrals by the functional Nörlund

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¹ Throughout the paper C denotes a positive constant, not necessarily the same at each occurrence.

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methods up to the present has not been studied to the same degree. In fact a recent paper of Bořcun [1] seems to be the only one which takes up the problem in a special direction and proves the following

THEOREM A. *If*

$$(2.2) \quad \psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\} \in BV(0, \infty),$$

$$(2.3) \quad \int_0^\infty \frac{|\psi(t)|}{t} dt < \infty,$$

$p(t)$ is a positive, differentiable, monotonic decreasing function such that $P(y) \rightarrow \infty$, for $y \rightarrow \infty$, and

$$(2.4) \quad X(y) \equiv yp(y)/P(y) \in BV(0, \infty),$$

$$(2.5) \quad S(y) = \frac{1}{P(y)} \int_0^y \frac{P(t)}{t} dt \in BV(0, \infty),$$

then the allied integral of the function $f(t)$ is $|N, p(y)|$ summable at the point $t=x$.

Write $V(y) = \int_y^\infty du/uP(u)$.

We observe that for a nonnegative function $p(t)$

$$(2.6) \quad \int_0^y \frac{P(u)}{u} du \leq CP(y),$$

if and only if

$$(2.7) \quad P(y)V(y) \leq C.$$

For if (2.6) holds, then

$$\begin{aligned} P(y) \int_y^\infty \frac{du}{uP(u)} &= 2P(y) \int_y^\infty \frac{p(u)}{P^2(u)} du \int_0^u \frac{P(t)}{t} dt - \frac{1}{P(y)} \int_0^y \frac{P(t)}{t} dt \\ &\leq CP(y) \int_y^\infty \frac{p(u)}{P^2(u)} du \leq C, \end{aligned}$$

that is, (2.7) holds. Conversely if (2.7) holds, then

$$\begin{aligned} \frac{1}{P(y)} \int_0^y \frac{P(u)}{u} du &= -\frac{1}{P(y)} \int_0^y P^2(u)V'(u) du \\ &= \frac{2}{P(y)} \int_0^y P(u)V(u)p(u) du - P(y)V(y) \\ &\leq C, \end{aligned}$$

and (2.6) follows.

We further note that (2.4) and (2.7) imply (2.5). For

$$\begin{aligned} S'(y) &= \frac{1}{y} - \frac{p(y)}{P^2(y)} \int_0^y \frac{P(t)}{t} dt \\ &= \frac{1}{yP^2(y)} \int_0^y \frac{P(y)P(t)(X(t) - X(y))}{t} dt \\ &= -\frac{1}{yP(y)} \int_0^y \frac{P(t)}{t} dt \int_t^y dX(u), \end{aligned}$$

we have

$$\begin{aligned} \int_0^\infty |S'(y)| dy &\leq \int_0^\infty \frac{dy}{yP(y)} \int_0^y \frac{P(t)}{t} dt \int_t^y |dX(u)| \\ &= \int_0^\infty \frac{dy}{yP(y)} \int_0^y |dX(u)| \int_0^u \frac{P(t)}{t} dt \\ &\leq C \int_0^\infty \frac{dy}{yP(y)} \int_0^y P(u) |dX(u)| \\ &\quad \text{(by virtue of the equivalence of (2.6) and (2.7))} \\ &= C \int_0^\infty P(u) |dX(u)| \int_u^\infty \frac{dy}{yP(y)} \leq C \int_0^\infty |dX(u)| \leq C. \end{aligned}$$

In view of the above discussions it is clear that Theorem A can be restated in the following equivalent form:

THEOREM B. If $\psi(t) \in BV(0, \infty)$, and $\int_0^\infty |\psi(t)|/t dt < \infty$, $p(t)$ is a positive, differentiable, monotonic decreasing function such that $P(y) \rightarrow \infty$ as $y \rightarrow \infty$ and

$$X(y) \equiv \frac{yp(y)}{P(y)} \in BV(0, \infty), \quad P(y) \int_y^\infty \frac{du}{uP(u)} \leq C,$$

then the allied integral of the function $f(t)$ is $|N, p(y)|$ summable at the point $t=x$.

In this paper we generalise the above theorem in the form of the following

THEOREM. Let $\varepsilon(t)$ be a positive, differentiable, monotonic nondecreasing function such that $\varepsilon(t)/t$ monotonically decreases as t increases to ∞ . Let $\psi(t)$, tending to a limit as $t \rightarrow \infty$, be a function of bounded variation in every finite interval $(0, a)$ ($a > 0$) such that

$$\int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| < \infty \quad (k \geq \pi)$$

and

$$\int_0^\infty \varepsilon\left(\frac{k}{t}\right) \frac{|\psi(t)|}{t} dt < \infty.$$

Further let $p(t)$ be a positive, differentiable and monotonic decreasing function such that $P(y) \int_y^\infty \varepsilon(u)/u P(u) du \leq C\varepsilon(y)$. Then the integral

$$\frac{1}{\pi} \int_0^\infty \varepsilon(u) du \int_{-\infty}^\infty f(t) \sin(t-x) u dt$$

is summable $|N, p(y)|$ at the point $t=x$.

A consequence of the above theorem is that Theorems A and B hold without condition (2.4).

3. We need the following lemmas for the proof of our theorem.

LEMMA 1 [1]. Let $v(u)$ be a nonnegative, monotonic decreasing function defined in $[0, \infty)$, differentiable in $[a, b]$, $0 \leq a < b \leq \infty$, and integrable in $[0, t]$ for any $t > 0$. Then for any positive t and y we have

$$\left| \int_a^b v(u) e^{i(y-u)t} du \right| \leq CV \left(\frac{1}{t} \right),$$

where $V(1/t) = \int_0^{1/t} v(u) du$.

The above lemma is the analogue for functions of a result for series given earlier by McFadden [2].

LEMMA 2. If $p(t)$ is a nonnegative and monotonic decreasing function such that

$$P(y) \int_y^\infty \frac{\varepsilon(u)}{u P(u)} du \leq C\varepsilon(y)$$

then

$$y \int_y^\infty \frac{\varepsilon(u)}{u^2} du \leq C\varepsilon(y).$$

PROOF. Since $P(u)/u$ is monotonic decreasing we have

$$y \int_y^\infty \frac{\varepsilon(u)}{u^2} du \leq P(y) \int_y^\infty \frac{\varepsilon(u)}{u P(u)} du \leq C\varepsilon(y),$$

and this completes the proof of the lemma.

4. Proof of the theorem. We have

$$(4.1) \quad \int_0^\infty \psi(t) \sin ut dt = \int_0^{\pi/2u} \psi(t) \sin ut dt + \frac{1}{u} \int_{\pi/2u}^\infty \cos ut d\psi(t)$$

since $\lim_{t \rightarrow \infty} \psi(t) = 0$, which follows under the hypotheses of the theorem. For, let $\lim_{t \rightarrow \infty} \psi(t) = l$, where l is a finite number. If $l \neq 0$, then for sufficiently large $t > t_0$, $|\psi(t)| > |l|/2$, and therefore

$$\int_{t_0}^{\infty} |\psi(t)| dt > \frac{|l|}{2} \int_{t_0}^{\infty} dt = +\infty$$

which contradicts the Lebesgue integrability of $f(t)$ over $(-\infty, \infty)$, and hence $\lim_{t \rightarrow \infty} \psi(t) = 0$.

Now

$$\begin{aligned} \sigma(y) &= \frac{1}{\pi P(y)} \int_0^y P(y-u) \varepsilon(u) du \int_{-\infty}^{\infty} f(t) \sin(t-x) u dt \\ &= \frac{2}{\pi P(y)} \int_0^y P(y-u) \varepsilon(u) du \int_0^{\infty} \psi(t) \sin ut dt, \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^{\infty} |\sigma'(y)| dy \\ \leq \frac{2}{\pi} \int_0^{\infty} \left| \int_0^y \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} \varepsilon(u) du \int_0^{\infty} \psi(t) \sin ut dt \right| dy. \end{aligned}$$

By virtue of (4.1) and the above inequality we have

(4.2)

$$\begin{aligned} \int_0^{\infty} |\sigma'(y)| dy \\ \leq \frac{2}{\pi} \int_0^{\infty} t |\psi(t)| dt \int_0^{\pi/2t} u \varepsilon(u) du \int_u^{\infty} \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} dy \\ + \frac{2}{\pi} \int_0^{\infty} |d\psi(t)| \int_{\pi/2t}^{\pi/t} dy \left| \int_{\pi/2t}^y \frac{p(y-u)P(y) - P(y-u)p(y)}{uP^2(y)} \varepsilon(u) \cos ut du \right| \\ + \frac{2}{\pi} \int_0^{\infty} |d\psi(t)| \int_{\pi/t}^{\infty} dy \left| \int_{\pi/2t}^{y/2} \frac{p(y-u)P(y) - P(y-u)p(y)}{uP^2(y)} \varepsilon(u) \cos ut du \right| \\ + \frac{2}{\pi} \int_0^{\infty} |d\psi(t)| \int_{\pi/t}^{\infty} \frac{dy}{P(y)} \left| \int_{y/2}^y \frac{\varepsilon(u)}{u} p(y-u) \cos ut du \right| \\ + \frac{2}{\pi} \int_0^{\infty} |d\psi(t)| \int_{\pi/t}^{\infty} \frac{p(y)}{P^2(y)} dy \left| \int_{y/2}^y \frac{\varepsilon(u)}{u} P(y-u) \cos ut du \right| \\ = I_1 + I_2 + I_3 + I_4 + I_5, \quad \text{say.} \end{aligned}$$

Clearly

(4.3)

$$I_1 = \frac{2}{\pi} \int_0^\infty t |\psi(t)| dt \int_0^{\pi/2t} u \varepsilon(u) du \int_u^\infty \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} dy$$

$$\leq C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) \frac{|\psi(t)|}{t} dt \leq C.$$

And

(4.4)

$$I_2 \leq \frac{2}{\pi} \int_0^\infty |d\psi(t)| \int_{\pi/2t}^{\pi/t} dy \int_{\pi/2t}^y \varepsilon(u) \frac{p(y-u)P(y) - P(y-u)p(y)}{uP^2(y)} du$$

$$= \frac{2}{\pi} \int_0^\infty |d\psi(t)| \int_{\pi/2t}^{\pi/t} \frac{\varepsilon(u)}{u} du \int_u^{\pi/t} \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} dy$$

$$\leq C \int_0^\infty t \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \int_{\pi/2t}^{\pi/t} du$$

$$\leq C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \leq C.$$

Again

(4.5)

$$I_3 \leq \frac{2}{\pi} \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty dy \left| \int_{\pi/2t}^{y/2} \sin ut \right.$$

$$\times \frac{\partial}{\partial u} \left\{ \frac{\varepsilon(u)}{u} \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} \right\} du \Big|$$

$$+ \frac{4}{\pi} \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty \varepsilon\left(\frac{y}{2}\right) \left| \frac{p(y/2)P(y) - P(y/2)p(y)}{yP^2(y)} \sin \frac{yt}{2} \right| dy$$

$$+ \frac{4}{\pi^2} \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \int_{\pi/t}^\infty \frac{p(y-\pi/2t)P(y) - P(y-\pi/2t)p(y)}{P^2(y)} dy$$

$$\leq C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty dy \int_{\pi/2t}^{y/2} \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} \left| \frac{d}{du} \left(\frac{\varepsilon(u)}{u} \right) \right| du$$

$$+ C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty dy \int_{\pi/2t}^{y/2} \frac{\varepsilon(u)}{u} \left| \frac{p'(y-u)P(y) - p(y-u)p'(y)}{P^2(y)} \right| du$$

$$+ C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty \frac{\varepsilon(y)}{y^2} dy$$

$$+ C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) \left(1 - \frac{P(\pi/2t)}{P(\pi/t)} \right) |d\psi(t)|$$

$$\begin{aligned}
&\leq C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/2t}^\infty \left| \frac{d}{du} \left(\frac{\varepsilon(u)}{u} \right) \right| du \int_{2u}^\infty \frac{p(y-u)P(y) - P(y-u)p(y)}{P^2(y)} dy \\
&\quad + C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/2t}^\infty \frac{\varepsilon(u)}{u} du \int_{2u}^\infty \left| \frac{p'(y-u)P(y) - p(y-u)p'(y)}{P^2(y)} \right| dy \\
&\quad + C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \\
&\leq C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| + C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/2t}^\infty \frac{\varepsilon(u)p(u)}{uP(2u)} du \\
&\leq C + C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/2t}^\infty \frac{\varepsilon(u)}{u^2} du \\
&\leq C + C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \leq C,
\end{aligned}$$

using Lemma 2 and the fact that $yp(y) \leq P(y)$.

By the application of the second mean value theorem and Lemma 1 we have

(4.6)

$$\begin{aligned}
I_4 &= \frac{4}{\pi} \int_0^\infty |d\psi(t)| \int_{\pi/t}^\infty \frac{\varepsilon(y/2)}{yP(y)} dy \left| \int_{y/2}^\xi p(y-u) \cos ut du \right| \quad \left(\frac{y}{2} < \xi < y \right) \\
&\leq C \int_0^\infty P\left(\frac{1}{t}\right) |d\psi(t)| \int_{\pi/t}^\infty \frac{\varepsilon(y)}{yP(y)} dy \\
&\leq C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \leq C.
\end{aligned}$$

Finally by the application of the second mean value theorem and Lemma 2 we have

(4.7)

$$\begin{aligned}
I_5 &= \frac{4}{\pi} \int_0^\infty |d\psi(t)| \int_{\pi/t}^\infty \frac{\varepsilon(y/2)p(y)P(y/2)}{yP^2(y)} dy \left| \int_{y/2}^\eta \cos ut du \right| \quad \left(\frac{y}{2} < \eta < y \right) \\
&\leq C \int_0^\infty \frac{|d\psi(t)|}{t} \int_{\pi/t}^\infty \frac{\varepsilon(y)}{y^2} dy \\
&\leq C \int_0^\infty \varepsilon\left(\frac{k}{t}\right) |d\psi(t)| \leq C.
\end{aligned}$$

Combining the estimates in (4.3) through (4.7) we get, in view of (4.2), that $\int_0^\infty |\sigma'(y)| dy \leq C$, and this completes the proof of the theorem.

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