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THE H_p-PROBLEM FOR GROUPS WITH CERTAIN CENTRAL FACTORS CYCLIC

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ABSTRACT. Let G be a group and $Hp(G)$ the subgroup generated by the elements of G of order different from p. Hughes conjectured that if $G>Hp(G)>1$, then $|G:Hp(G)|=p$. In this paper it is shown that if G is a finite p-group and certain central factors of G are cyclic or if the normal subgroups of G of a certain order are two generated, then the Hughes conjecture is true for G.

1. **Introduction.** Let G be a group and $H_p(G)$ the subgroup of G generated by the elements of order different from p. Hughes **[6]** conjectured that if $G>H_p(G)>1$, then $|G:H_p(G)|=p$. Although Wall **[11]** has shown the conjecture is false for $p=5$, the conjecture is true in the following cases: $p=2$ **[5]**, $p=3$ **[10]**, regular p-groups **[4]**, finite groups which are not p-groups **[7]**, finite metabelian p-groups **[4]** (see also **[9]**, p. 42]), finite p-groups with class at most $2p-2$ **[9]**, finite p-groups with the property that every 3-generated subgroup has class at most p **[3]** and finite p-groups with cyclic lower central factors **[4]**. In this paper it is shown that if G is a finite p-group and certain central factors of G are cyclic or if the normal subgroups of G of a certain order are two generated, then the Hughes conjecture is true for G.

2. **Notation.** If G is a group of nilpotence class c we use $G=L_1(G)>L_2(G)>\cdots>L_{c+1}(G)=1$ to denote the lower central series of G and $1=Z_0(G)<Z_1(G)<\cdots<Z_c(G)=G$ to denote the upper central series of G. The exponent of G is denoted by $\exp(G)$ and we say G is an *ECF*-group if $G/L_2(G)$ has exponent p and $L_i(G)/L_{i+1}(G)$ is cyclic for all $i\geqq 2$.

3. THEOREM 1. *Let G be a finite p-group of class $c>p>2$ with $H_p(G)\neq 1$ and $L_i(G)/L_{i+1}(G)$ cyclic for $i=2,\cdots,p$. Then $|G:H_p(G)|\leqq p$.*

PROOF. Assume G satisfies the hypothesis of the theorem and has minimum order such that $|G:H_p(G)|>p$. Let y belong to $H_p(G)$ with $|y|>p$. Then for any element z of order p in $L_c(G)=L_c$ we have $(yz)^p\neq 1$

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so that yz belongs to $H_p(G)$ and it follows that $L_c \leqq H_p(G)$. Thus $H_p(G/L_c) \leqq H_p(G)/L_c$ and $|G/L_c : H_p(G/L_c)| \geqq |G : H_p(G)| > p$. Since $L_i(G/L_c)/L_{i+1}(G/L_c)$ is isomorphic to $L_i(G)/L_{i+1}(G)$ for $i=2, \cdots, p$, G/L_c satisfies the latter portion of the hypothesis. By the result of Macdonald [**9**] we may assume $c > 2p-2$ so that $c-1 > p$. It follows that $H_p(G/L_c) = 1$ and $\exp(G/L_c) = p$. Since

$$p = \exp(G/L_c) \geqq \exp(L_i(G)/L_{i+1}(G))$$

for $i=2, \cdots, c$, we have $|L_i(G)/L_{i+1}(G)| = p$ for $i=2, \cdots, p$ and a theorem of Blackburn [**1**, p. 74] yields $|L_i(G)/L_{i+1}(G)| = p$ for $i=2, \cdots, c$. There are now several ways to arrive at a contradiction. One is directly obtained by applying the result of Hogan and Kappe [**4**] mentioned in the introduction. Other ways to obtain a contradiction are to use Corollary 1 in [**1**, p. 69] or to apply Lemma 2 in [**4**] to G/L_c together with the theorem of Macdonald [**9**].

THEOREM 2. *Let G be a finite p-group of class $c > p > 2$ with $H_p(G) \neq 1$ and $Z_{i+1}(G)/Z_i(G)$ cyclic for $i=p-2, \cdots, c-2$. Then $|G : H_p(G)| \leqq p$.*

PROOF. Assume G satisfies the hypothesis of the theorem and has minimum order such that $|G : H_p(G)| > p$. As in the proof of Theorem 1, we have $|G/Z_1 : H_p(G/Z_1)| > p$. Since $Z_{i+1}(G/Z_1)/Z_i(G/Z_1)$ is isomorphic to $Z_{i+2}(G)/Z_{i+1}(G)$ for $i=1, \cdots, c-3$ and G/Z_1 has class $c-1$, G/Z_1 satisfies the latter portion of the hypothesis. As in the proof of Theorem 1 we may assume $H_p(G/Z_1) = 1$ and therefore $\exp(G/Z_1) = p$. Since

$$p = \exp(G/Z_1) \geqq \exp(Z_{i+1}(G)/Z_i(G)) \quad \text{for } i = 1, 2, \cdots, c-2,$$

we have $|Z_{i+1}(G)/Z_i(G)| = p$ for $i=p-2, \cdots, c-2$. We let $Z_{p-2}(G) = Z_{p-2}$ and consider G/Z_{p-2}. Since $Z_i(G/Z_{p-2}) = Z_{p-2+i}(G)/Z_{p-2}(G)$ it follows that $|Z_{i+1}(G/Z_{p-2})/Z_i(G/Z_{p-2})| = p$ for $i=0, \cdots, c-p$. Thus

$$L_i(G/Z_{p-2}) = Z_{c-p+3-i}(G/Z_{p-2}) \quad \text{for } i = 1, \cdots, c-p+3$$

and therefore G/Z_{p-2} is an *ECF*-group of exponent p and class $c-p+2$. Since Hogan and Kappe [**4**] have shown that an *ECF*-group of exponent p has class at most p, it follows that $c \leqq 2p-2$ and the result of Macdonald [**9**] gives a contradiction.

We remark that the restriction that $c > p > 2$ in Theorems 1 and 2 is merely a technicality in view of the results mentioned in the introduction.

THEOREM 3. *Suppose $|G| = p^n$ with $H_p(G) \neq 1$ and that for some fixed r with $3 \leqq r \leqq n-1$ every normal subgroup of order p^r has two generators Then $|G : H_p(G)| \leqq p$.*

PROOF. *Case* 1. $r=n-1$. Assume G satisfies the hypothesis of the theorem and has minimum order such that $|G:H_p(G)|>p$. Let c denote the class of G and let N be a subgroup of $L_c(G)$ of order p. Clearly we may assume $n\geqq 6$. Since every normal subgroup of G/N of order p^{n-2} has two generators, it follows that $\exp(G/N)=p$ and therefore $P_1(G)=\langle g^p|g\in G\rangle\leqq N\leqq L_c(G)$. By the results mentioned in the introduction we may assume that $c>p>2$. From a theorem of Blackburn [**2**, p. 19] we see that G is either metacyclic or $P_1(G)=L_3(G)$ and in either instance we clearly have a contradiction.

Case 2. $3\leqq r\leqq n-2$. The known results imply that we may assume $c\geqq p\geqq 5$ and $n\geqq 5$. According to a theorem of Blackburn [**2**, p. 16] one of the following is true:

(i) G is metacyclic,

(ii) G is a 3-group of maximal class,

(iii) the elements of G of order at most p form a normal subgroup E of order p^3 and G/E is cyclic.

Clearly we need only discuss case (iii). Since $|E|=p^3$ implies $E\leqq Z_3(G)\leqq Z_{c-1}(G)$ [**8**, p. 301], it follows that G/Z_{c-1} is cyclic. But G/Z_{c-1} is cyclic only when G is cyclic. This contradiction completes the proof.

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