PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

FIXED POINTS OF CERTAIN SELF MAPS ON AN INTERVAL

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ABSTRACT. Let T be a self map on a bounded interval $[a, b]$ with $a, b \in T([a, b])$. Suppose that for any x, y in $[a, b]$,

$$|T(x) - T(y)| \leqq \tfrac{1}{2}(|x - T(x)| + |y - T(y)|).$$

It is shown without the continuity of T that the midpoint of $[a, b]$ is a fixed point of T. A nontrivial example is given.

1. Main theorem.

THEOREM. *Let T be a self map on a bounded closed interval $[a, b]$ with $a, b \in T([a, b])$. Suppose that for all x, y in $[a, b]$,*

$$|T(x) - T(y)| \leqq (|x - T(x)| + |y - T(y)|)/2. \tag{1}$$

Then the midpoint of $[a, b]$ is the unique fixed point of T.

PROOF. Since $b-a$ is uniquely maximal for $|x-y|$, $T(a)=b$ and $T(b)=a$. Let c be the midpoint $(a+b)/2$ of $[a, b]$. Then $|T(b)-T(c)|$ and $|T(a)-T(c)|$ are each equal to $|a-c|=|b-c|=(b-a)/2$. Hence $T(c)=c$. From (1), c is the unique fixed point of T.

We note here that if T is a continuous self map on a bounded closed interval $Y=[a, b]$ which satisfies (1), then $X=\bigcap_{i=1}^{\infty} T^i(Y)$ is nonempty, compact and connected, and $T(X)=X$; hence the midpoint of X is a fixed point of T. Comparing this result with that of R. L. Franks and R. P. Marzec [1], we remark here that for any x in Y and for any t in $(0, 1)$, $\{T_t^n(x)\}$ converges to a fixed point of T, where $T_t(z)=(1-t)z+tT(z)$ for all z in Y. This result was proved in [2] with a much more general setting.

2. **Examples.** Let T be a function on a bounded interval X into the real line. T is *nonexpanding at* x in X if $|T(x)-T(y)|\leqq|x-y|$ for all y in X. T is *nonexpanding* if it is nonexpanding at every point in X. If

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Received by the editors November 10, 1972 and, in revised form, April 10, 1973.

AMS (MOS) subject classifications (1970). Primary 47H10; Secondary 54H25.

[1] This research was partially supported by the National Research Council of Canada Grant A8518 and a grant from the Canadian Mathematical Congress; it was prepared at the Summer Research Institute, Department of Mathematics, University of Alberta. The author wishes to thank the referee for his suggestions.

T satisfies the conditions of our main theorem, then T is nonexpanding at $x=(a+b)/2$ and for all x in $[a, b]$, $|T^2(x)-T(x)|\leqq|T(x)-x|$. These are key points in constructing examples.

EXAMPLE 1. Let T be the self map on $[0, 1]$ defined by

$$T(0) = 1, \quad T(1) = 0, \quad T(x) = \tfrac{1}{2}, \quad \text{for all } x \text{ in } (0, 1).$$

Then T is nonexpanding only at $x=\frac{1}{2}$ and satisfies the conditions of our main theorem. Although T is not nonexpanding at any x in $(0, 1)$ other than $\frac{1}{2}$, T, restricted to $(0, 1)$, is a nonexpanding self map on $(0, 1)$.

When T is required to be continuous, the example cannot be so trivial. Indeed, if T is a continuous self map on $[0, 1]$ and if $0, 1 \in T([0, 1])$, then by connectedness of $T([0, 1])$, $T([0, 1])=[0, 1]$.

EXAMPLE 2. Let T be the self map on $[0, 1]$ defined by

$$\begin{aligned} T(x) &= 1 - 2x + 2x^2 && \text{if } x \leqq \tfrac{1}{2}, \\ T(x) &= 2x(1 - x) && \text{if } x > \tfrac{1}{2}. \end{aligned}$$

Then: (a) T is a bijective continuous self map on $[0, 1]$ and therefore is a homeomorphism of $[0, 1]$ onto $[0, 1]$. (b) T is differentiable on $(0, 1)$ and $|T'(x)|\leqq 1$ for all x in $[\frac{1}{4}, \frac{3}{4}]$. So by the mean value theorem, $|T(x)-T(y)|\leqq|x-y|$ for all x, y in $[\frac{1}{4}, \frac{3}{4}]$. However T is nonexpanding only at $x=\frac{1}{2}$ and T, restricted to $[\frac{1}{4}, \frac{3}{4}]$, is nonexpanding but is not a self map on $[\frac{1}{4}, \frac{3}{4}]$. (c) T satisfies the conditions of our main theorem. (d) Let $t \in (0, \frac{1}{3})$. Let T_t be the self map on $[0, 1]$ defined by

$$T_t(z) = (1 - t)z + tT(z), \qquad z \in [0, 1].$$

Then $x=\frac{1}{2}$ is a fixed point of T_t. However, T_t does not satisfy the condition (1) in our main theorem:

$$|T_t(0) - T_t(\tfrac{1}{2})| = \tfrac{1}{2} - t$$

and

$$(|0 - T_t(0)| + |\tfrac{1}{2} - T_t(\tfrac{1}{2})|)/2 = t/2 < \tfrac{1}{2} - t.$$

The above argument and therefore conclusions hold for any T which satisfies the conditions of our main theorem with $a=0$ and $b=1$.

REFERENCES

1. R. L. Franks and R. P. Marzec, *A theorem on mean-value iterations*, Proc. Amer. Math. Soc. **30** (1971), 324–326. MR **43** #6375.

2. Chi Song Wong, *Approximations to fixed points of generalized nonexpansive mappings* (to appear).

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