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ON THE MARX CONJECTURE FOR THE CONVEX HULLS OF FAMILIES OF STARLIKE AND CONVEX MAPPINGS

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ABSTRACT. We prove a Marx conjecture for the closed convex hull of the family of functions which are starlike of order α and k-fold symmetric. We obtain precise results for the functions which are starlike, starlike of order $\frac{1}{2}$, and starlike with 2-fold symmetry in their power series expansions.

Introduction. Let Δ denote the unit disk $\{z:|z|<1\}$ and let A denote the set of functions analytic in Δ. When A is given the topology of uniform convergence on compact subsets of Δ it is known [9, p. 150] to be a locally convex linear topological space. We recall the definition of subordination between two functions f and g analytic in Δ. We say f is subordinate to g, denoted $f \prec g$, if there exists an analytic function $\phi(z)$ so that $\phi(0)=0$, $|\phi(z)|<1$, and $f(z)=g(\phi(z))$ for z in Δ. We let X denote the unit circle $\{x:|x|=1\}$ and $\mathscr{P}$ denote the set of probability measures on X.

We consider the class of functions denoted by $\mathrm{St}_k(\alpha)$ which are k-fold symmetric and starlike of order α. We recall that $f(z)$ is in $\mathrm{St}_k(\alpha)$ if and only if

$$f(z)=\sum_{m=0}^{\infty} a_{mk+1} z^{mk+1} \quad \text{and} \quad \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$$

where $\alpha<1$, $k=1,2,\cdots$, and z is in Δ. The functions $\mathrm{St}_1(\alpha)$ were introduced in [7] by M. S. Robertson.

In [1] L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken determined the closed convex hull of $\mathrm{St}_k(\alpha)$ denoted by $\mathscr{H}\,\mathrm{St}_k(\alpha)$ to be the set of functions

$$\left\{f : f(z)=\int_X \frac{z}{(1-xz^k)^{(2-2\alpha)/k}}\, d\mu(x) \text{ and } \mu \in \mathscr{P}\right\}.$$

In 1932 in [4, p. 66] A. Marx conjectured that if $f \in \mathrm{St}_1(0)$ then the range of $f'(z) \subset$ range of $(z/(1-z)^2)'$ for all z in Δ. In [3] J. A. Hummel

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proved this conjecture to be false. In [5] R. McLaughlin proved that if $f\in \mathrm{St}_1(\alpha)$ then there exists a radius denoted by $r(\alpha)$ so that $0<r(\alpha)\leqq 1$ and the range of $f'(z)$ is contained in the range of $[z/(1-z)^{2-2\alpha}]'$ for $|z|<r(\alpha)$. The numbers $r(\alpha)$ were computed as the roots of a certain polynomial.

In Theorem (1) we prove that if $f\in \mathscr{H}\ \mathrm{St}_k(\alpha)$ then there exists a radius denoted by $r(\alpha, k)$ such that $0<r(\alpha, k)\leqq 1$ and the range of $f'(z)$ is contained in the range of $[z/(1-z^k)^{(2-2\alpha)/k}]'$ for $|z|\leqq r(\alpha, k)$. For some special cases of interest, we compute the exact value of $r(\alpha, k)$. We also consider the class of univalent convex mappings denoted by K.

1. The Marx conjecture for $\mathscr{H}\ \mathrm{St}_k(\alpha)$.

THEOREM (1). *If $f\in\mathscr{H}\ \mathrm{St}_k(\alpha)$, then there exists a positive number denoted by $r(\alpha, k)$ such that the range of $f'(z)$ lies in the range of $[z/(1-z^k)^{(2-2\alpha)/k}]'$ for $|z|\leqq r(\alpha, k)$. The result is sharp.*

PROOF. It was proven in [1] that when $f\in\mathscr{H}\ \mathrm{St}_k(\alpha)$ then

$$f(z)=\int_X \frac{z}{(1-xz^k)^{(2-2\alpha)/k}}\,d\mu(x)$$

for some $\mu\in\mathscr{P}$. We see by a short computation that

$$f'(z)=\int_X\left[\frac{z}{(1-xz^k)^{(2-2\alpha)/k}}\right]' d\mu(x)=\int_X p'(xz)\,d\mu(x)$$

where $p(z)=z/(1-z^k)^{(2-2\alpha)/k}$. We will prove that $p'(z)$ has a positive radius of convexity denoted by $r(\alpha, k)$. The range containment will then follow, since the integral can be approximated by sums of the form

$$\sum_{i=1}^n \lambda_i\left[\frac{z}{(1-x_iz^k)^{(2-2\alpha)/k}}\right]'$$

where $\lambda_i\geqq 0$ and $\sum_{i=1}^n \lambda_i=1$. Since $p(z)=z+\cdots$ satisfies the normalizations $p(0)=0$ and $p'(0)=1$, it is easy to verify that $p'(z)$ has a positive radius of convexity which we denote by $r(\alpha, k)$.

Suppose that $|z|>r(\alpha, k)$. The radius of univalence of $p(z)$ is strictly larger than $r(\alpha, k)$. Therefore, there exist two distinct points $re^{i\theta_1}$ and $re^{i\theta_2}$ where $|z|=r$ so that

$$w=\frac{1}{2}\left[\frac{re^{i\theta_1}}{(1-xr^ke^{ki\theta_1})^{(2-2\alpha)/k}}\right]'+\frac{1}{2}\left[\frac{re^{i\theta_2}}{(1-xr^ke^{ki\theta_2})^{(2-2\alpha)/k}}\right]'$$

is not the image of $|z|\leqq r$ under $p'(z)$. To each number $z=re^{i\theta}$ chosen so that $|z|=r>r(\alpha, k)$ and r is sufficiently close to $r(\alpha, k)$ we may choose μ to be a measure with mass $\frac{1}{2}$ at each of the points $x_1=e^{i(\theta_1-\theta)}$ and

$x_2 = e^{i(\theta_2 - \theta)}$. Then we see that

$$g(z) = \int_X \frac{z}{(1 - xz^k)^{(2-2\alpha)/k}} \, d\mu(x)$$

is in $\mathscr{H}\,\mathrm{St}_k(\alpha)$ but $g'(z) = w$ is not in the image of $|z| \leqq r$ under $p'(z)$. Hence, the result is sharp.

REMARKS. (1) Since $\mathrm{St}_k(\alpha) \subset \mathscr{H}\,\mathrm{St}_k(\alpha)$, it is clear that we have proven a result for the class $\mathrm{St}_k(\alpha)$.

(2) A more detailed argument shows that $r(\alpha, 1) \to 1$ as $\alpha \to 1$ and $r(0, k) \to 1$ as $k \to \infty$.

(3) When $k=1$ and $\alpha=0$ it is known [**8**, p. 33] that the radius of convexity of $[z/(1-z)^2]'$ is $2-\sqrt{3}$ and hence $r(0, 1) = 2-\sqrt{3}$. So we have a sharp form of the Marx conjecture for $\mathscr{H}\,\mathrm{St}_1(0)$.

(4) Theorem (3) in [**1**] contains the result when $\alpha = \frac{1}{2}$ and $k=1$ that $\mathscr{H}\,\mathrm{St}_1(\frac{1}{2})$ consists exactly of the functions found in [**2**, p. 94] to be $\mathscr{H}K$. It is easy to compute that $r(\frac{1}{2}, 1) = \frac{1}{2}$. We recall that the Marx conjecture for K [**4**, p. 62] and $\mathrm{St}_1(\frac{1}{2})$ [**6**, p. 278] is known to hold for the full unit disk.

2. The Marx conjecture for $\mathscr{H}\,\mathrm{St}_2(0)$.

LEMMA 1. *The function* $g(z) = (1+z^2)/(1-z^2)^2 = [z/(1-z^2)]'$ *is convex for* $|z| \leqq (4-\sqrt{13})^{1/2}$ *and bivalent in all of* Δ.

PROOF. Clearly, if $h(z) = (1+z)/(1-z)^2$ is convex and univalent for $|z| \leqq 4-\sqrt{13}$, then $g(z) = (1+z^2)/(1-z^2)^2$ will be convex and bivalent for $|z| \leqq (4-\sqrt{13})^{1/2}$. It is an easy matter to prove directly that $h(z)$ is univalent in all of Δ. We know that $h(z)$ is convex and univalent if and only if $\mathrm{Re}[1 + zh''(z)/h'(f)] \geqq 0$ since $h'(0) \neq 0$. A calculation shows that

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) = \mathrm{Re}\left(\frac{3 + 8z + z^2}{(1-z)(3+z)}\right).$$

The last expression is positive if and only if $\mathrm{Re}(3+8z+z^2)(1-\bar{z})(3+\bar{z}) \geqq 0$. Let $r = |z|$ and $x = \mathrm{Re}\, z = r\cos\theta$. The previous condition becomes

$$\begin{aligned}\mathrm{Re}\{9 - 16\,|z|^2 - |z|^4) - (6 + 8\,|z|^2)\bar{z} - 3(\bar{z})^2 + (24 - 2\,|z|^2) + 3z^2\} \\ = 9 - 16r^2 - r^4 + 18x - 10r^2x \geqq 0.\end{aligned}$$

So, we must decide when $p(x, r) = 9 - 16r^2 - r^4 + 18x - 10r^2x$ is positive. Since $x \geqq -r$ we have

$$\begin{aligned} p(x, r) \geqq p(-r, r) &= 9 - 18r - 16r^2 + 10r^3 - r^4 \\ &= (r+1)(r-3)(-r^2 + 8r - 3) \geqq 0 \end{aligned}$$

when $r \leqq 4-\sqrt{13}$ the smallest positive root of $p(-r, r)$.

THEOREM (2). *If $f\in\mathscr{H}\,\mathrm{St}_2(0)$, then the range of $f'(z)$ is contained in the range of $[z/(1-z^2)]'$ for $|z|\leqq(4-\sqrt{13})^{1/2}$. The result is sharp.*

PROOF. Use Lemma 1 and proceed as in the proof of Theorem (1) where $\alpha=0$ and $k=2$.

REMARKS. This result for $\mathscr{H}\,\mathrm{St}_2(0)$ and of course $\mathrm{St}_2(0)$ suggests the conjecture that if $f\in\mathrm{St}_2(0)$ then the range of $f'(z)\subset$ the range of $[z/(1-z^2)]'$ for z in Δ. We note that $r(0,2)=(4-\sqrt{13})^{1/2}$ is approximately 0.63. Recall that for $\mathscr{H}\,\mathrm{St}_1(\frac{1}{2})$ we found $r(\frac{1}{2},1)=\frac{1}{2}$ while for the class $\mathrm{St}_1(\frac{1}{2})$ the result held in the full disk Δ.

3. A Marx-like conjecture for $\mathscr{H}K$.

LEMMA 2. *The function $g(z)=z/(1-z)^3$ has a radius of convexity equal to $\frac{1}{8}(7-\sqrt{33})$ and a radius of univalence equal to $\frac{1}{2}$.*

PROOF. It is trivial to verify that the radius of univalence of $g(z)$ is $\frac{1}{2}$. The function $g(z)$ is convex and univalent for $|z|\leqq r$ if and only if

$$\mathrm{Re}\left[1+\frac{zg''(z)}{g'(z)}\right]\geqq 0$$

since $g'(0)\neq 0$. A short calculation shows that

$$\mathrm{Re}[1+zg''(z)/g'(z)]=\mathrm{Re}[(1+7z+4z^2)/(1-z)(1+2z)].$$

The last expression is positive if and only if

$$\mathrm{Re}(1+7z+4z^2)(1-\bar{z})(1+2\bar{z})\geqq 0.$$

Let $r=|z|$ and $x=\mathrm{Re}\,z=r\cos\theta$. The previous condition becomes

$$\begin{aligned}\mathrm{Re}(1+(7+4r^2)z+4z^2+(1-14r^2)\bar{z}-2(\bar{z})^2+7r^2-8r^4)&\\ =1+(8-10r^2)x+4x^2+5r^2-8r^4.&\end{aligned}$$

Consider $p(x,r)=1+(8-10r^2)x+4x^2+5r^2-8r^4$. It is easy to verify that

$$\partial p/\partial x=8-10r^2+8x\geqq 0\quad\text{for } r\leqq\tfrac{1}{2}\text{ and } x\geqq -r.$$

Hence, to minimize $p(x,r)$ we may set $x=-r$. We then have

$$p(-r,r)=2(1+r)(r-\tfrac{1}{2})(-1+7r-4r^2)$$

and it is simple to show $p(-r,r)\geqq 0$ for $r\leqq\frac{1}{8}(7-\sqrt{33})$.

THEOREM (3). *If $f\in\mathscr{H}K$, then the range of $zf''(z)$ is contained in the range of $z(z/(1-z))''$ for $|z|\leqq\frac{1}{8}(7-\sqrt{33})$. The result is sharp.*

PROOF. As mentioned in remark (4) above, we know that

$$f(z) = \int_X z/(1 - xz)\, d\mu(x)$$

for μ in $\mathscr{P}$. Hence $zf''(z) = \int_X 2xz/(1-xz)^3\, d\mu(x)$. By Lemma 2 and the type of arguments made in the proof of Theorem (1) the result follows.

REMARKS. (1) This result suggests the problem of finding the largest radius r such that if $f \in K$ then $zf''(z) \prec z(z/(1-z))''$ for $|z| \leqq r$. By the above result $r \geqq \frac{1}{8}(7-\sqrt{33})$.

(2) Since $\mathscr{H}\, \mathrm{St}_1(\frac{1}{2}) = \mathscr{H}K$ as mentioned in Remark (4) above, we actually have proven Theorem (3) for the classes K and $\mathrm{St}_1(\frac{1}{2})$.

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