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TWO RADIUS OF CONVEXITY PROBLEMS

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ABSTRACT. The sharp radius of convexity of functions with prescribed second coefficient is found for the two classes: functions starlike of order α, and functions whose derivative has real part greater than α.

1. **Introduction.** Let $\mathscr{P}(\alpha)$ denote the class of functions $P(z)=1+b_1z+\cdots$ which are analytic and satisfy $\operatorname{Re}\{P(z)\}>\alpha$ for $|z|<1$ where $\alpha\in[0,1)$. If $\varepsilon=\exp\{-i\arg b_1\}$ then $P(\varepsilon z)=1+|b_1|z+\cdots$ and we see that it is no actual restriction to limit our study of $\mathscr{P}(\alpha)$ to functions with a nonnegative real first coefficient. It is known [2] that $|b_1|\leqq 2(1-\alpha)$ and we define $\mathscr{P}_b(\alpha)=\{P(z)\in\mathscr{P}(\alpha):P'(0)=2b(1-\alpha)\}$ for $b\in[0,1]$. This paper extends results found in [1] by obtaining a lower bound on $\operatorname{Re}\{zP'(z)/P(z)\}$ for $P(z)\in\mathscr{P}_b(\alpha)$ and subsequently applying the results to obtain a sharp estimate for the radius of convexity of the two classes $\mathscr{S}_a^*(\alpha)$ and $\mathscr{P}_a'(\alpha)$ for each $a\in[0,1]$ and $\alpha\in[0,1)$ where

$$\mathscr{P}_a'(\alpha)=\{F(z)=z+a(1-\alpha)z^2+\cdots:F'(z)\in\mathscr{P}_a(\alpha)\}$$

and

$$\mathscr{S}_a^*(\alpha)=\{f(z)=z+2a(1-\alpha)z^2+\cdots:\operatorname{Re}\{zf'(z)/f(z)\}>\alpha\ \text{ for } |z|<1\}.$$

The technique used to obtain the results is based on a method of Singh and Goel [4] and extends some of the results found therein.

2. **Preliminaries.** Let $\mathscr{A}$ denote the class of functions $w(z)$ such that $w(0)=0$ which are also analytic and satisfy $|w(z)|<1$ for $|z|<1$. We will occasionally use $\beta=2\alpha-1$ to simplify computations and statements of results.

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LEMMA 1. *If* $P(z) \in \mathscr{P}_b(\alpha)$, *then* $|P(z)-A_b| \leqq D_b$ *for* $b \in [0, 1]$ *where*

$$A_b = \frac{(1+br)^2 - \beta(b+r)^2r^2}{(1+2br+r^2)(1-r^2)}, \qquad D_b = \frac{(1-\beta)(b+r)(1+br)r}{(1+2br+r^2)(1-r^2)},$$

$$\beta = 2\alpha - 1 \quad \textit{and} \quad r = |z| < 1.$$

PROOF. It is known that $P(z) \in \mathscr{P}_b(\alpha)$ if and only if there exists some $w(z) \in \mathscr{A}$ such that

$$P(z) = [1 + (1-2\alpha)w(z)]/[1 - w(z)]. \tag{1}$$

So $w(z) = [P(z)-1]/[P(z)+(1-2\alpha)] = bz + \cdots = z\varphi(z)$ where $\varphi(z)$ is analytic and $|\varphi(z)| \leqq 1$ for $|z|<1$ with $\varphi(0)=b$. Now, $(\varphi(z)-b)/(1-b\varphi(z)) \prec z$ and it follows that $\varphi(z) \prec (z+b)/(1+bz)$ and

$$|w(z)| = |z\varphi(z)| \leqq |z|\,(|z|+b)/(1+b\,|z|).$$

Let

$$g(z) = \frac{|z|+b}{1+b\,|z|}\,z \quad \text{and} \quad T(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

We note that the image of $|z| \leqq r$ under $g(z)$ is a disk and that $T(z)$ is a bilinear transformation. The image of $|z| \leqq r < 1$ under $P(z)$ is contained within the image of $|z| \leqq r$ under $(T \circ g)(z)$ which proves the lemma.

REMARK 1. For a fixed $r \in [0, 1)$, $A_b - D_b$ decreases as b increases over the interval $[0, 1]$ since

$$\frac{\partial}{\partial b}(A_b - D_b) = -\frac{(1-\alpha)(1-r^2)}{(1+2br+r^2)^2} < 0.$$

LEMMA 2. *Let* $P(z) \in \mathscr{P}_b(\alpha)$, $\beta = 2\alpha - 1 \in [-1, 1)$, *and* $k \geqq 1$; *then for* $b \in [0, 1]$

$$\operatorname{Re}\left\{kP(z) + \frac{\beta}{P(z)}\right\} - \frac{|z|^2\,|P(z)-\beta|^2 - |P(z)-1|^2}{(1-|z|^2)\,|P(z)|}$$

$$\geqq \frac{\begin{gathered}(k+\beta) + 2((k+2)\beta + k)br \\ + ((k+1)(\beta+1)^2b^2 + 2(k+\beta) - (\beta-1)^2)r^2 \\ + 2(k\beta + (k+2))\beta b r^3 + (1+k\beta)\beta r^4\end{gathered}}{(1+2br+r^2)(1+(\beta+1)br+\beta r^2)} \quad \text{if } R_b \geqq R' \tag{2}$$

$$\geqq 2((1+k)(1+\beta)A_1)^{1/2} - A_1 \quad \text{if } R_b \leqq R' \tag{3}$$

where A_b *and* D_b *are as in Lemma* 1 *with* $R_b = A_b - D_b$ *and*

$$R' = ((1+\beta)A_1/(1+k))^{1/2}.$$

PROOF. Let $P(z)=A_b+u+iv$ and $R^2=(A_b+u)^2+v^2$ with $r=|z|$, then

$$\operatorname{Re}\left\{kP(z)+\frac{\beta}{P(z)}\right\}-\frac{|z|^2\,|P(z)-\beta|^2-|P(z)-1|^2}{(1-|z|^2)\,|P(z)|}$$

$$(4)\qquad =\left(k-2\left(\frac{1-\beta r^2}{1-r^2}\right)R^{-1}+\beta R^{-2}\right)(A_b+u)$$

$$+R+\left(\frac{1-\beta^2r^2}{1-r^2}\right)R^{-1}.$$

Since $A_1=(1-\beta r^2)/(1-r^2)$ and $D_1=((1-\beta)r)/(1-r^2)$, the right-hand side of (4) may be written

$$(5)\qquad (k-2A_1R^{-1}+\beta R^{-2})(A_b+u)+R+(A_1^2-D_1^2)R^{-1}.$$

Let $S_b(u,v)$ denote the expression appearing in (5), then regrouping the terms we have

$$(6)\qquad \begin{aligned} S_b(u,v)&=(k+\beta R^{-2})(A_b+u)+(((A_b+u)-A_1)^2+v^2-D_1^2)R^{-1},\\ \partial S_b/\partial v&=vR^{-4}T_b(u,v),\end{aligned}$$

$$(7)\qquad \begin{aligned} T_b(u,v)&=-2\beta(A_b+u)+(D_1^2-(A_b+u-A_1)^2-v^2)R+2R^3\\ &=-2\beta(A_b+u)+(D_1^2+2A_1(A_b+u)-A_1^2)R+R^3.\end{aligned}$$

Denote by $F_b(R)$ the right-hand side of (7) with $R\cos\psi=A_b+u$; then

$$(8)\qquad F_b(R)=2(A_1R-\beta)R\cos\psi+(D_1^2-A_1^2+R^2)R.$$

Geometrical considerations show that for $R\in[A_b-D_b,A_b+D_b]$ the function $F_b(R)$ increases with increasing R. Since $R\cos\psi$ is the projection onto the real axis, it must lie on the diameter of the circle of Lemma 1; we have $R\cos\psi\geqq A_1-D_1$ by virtue of Remark 1 and so for all $b\in[0,1]$

$$\begin{aligned} F_b(R)&\geqq(2(A_1(A_1-D_1)-\beta)+(D_1^2-A_1^2+(A_1-D_1)^2))(A_1-D_1)\\ &=\left((1-\beta)\left(\frac{1-\beta r^2}{(1+r)^2}\right)\right)(A_1-D_1)>0.\end{aligned}$$

Hence the minimum of $S_b(u,v)$ inside the circle $|P(z)-A_b|\leqq D_b$ is attained on the diameter. Setting $v=0$ in (5) we obtain

$$(9)\qquad L_b(R)=(1+k)R+(\beta+1)A_1R^{-1}-2A_1$$

where $R=A_b+u\in[A_b-D_b,A_b+D_b]$. The absolute minimum of $L_b(R)$ in $(0,\infty)$ is attained at

$$(10)\qquad R'=((1+\beta)A_1/(1+k))^{1/2}$$

and yields (3). Clearly $R' < A_b + D_b$, but it is not always true that $R' \in [A_b - D_b, A_b + D_b]$. When R' is not in the interval then $L_b(R)$ achieves its minimum at the point $R_b = A_b - D_b$ from which we get (2). The transition from (2) to (3) takes place for those values of k and β for which $R' = R_b$.

LEMMA 3 [4]. *If $w(z) \in \mathscr{A}$, then for $|z|<1$*

$$|zw'(z) - w(z)| \leqq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}. \tag{11}$$

THEOREM 1. *Suppose $P(z) \in \mathscr{P}_b(\alpha)$ for $\alpha \in [0, 1)$, then, for all $b \in [0, 1]$,*

$$\operatorname{Re}\left\{\frac{zP'(z)}{P(z)}\right\} \geqq \frac{-2(1-\alpha)r}{1 + 2\alpha br + (2\alpha - 1)r^2} \frac{b + 2r + br^2}{1 + 2br + r^2} \quad \text{if } R' \leqq R_b \tag{12}$$

$$\geqq (2(\alpha A_1)^{1/2} - A_1 - \alpha)/(1-\alpha) \quad \text{if } R' \geqq R_b \tag{13}$$

where $R_b = A_b - D_b$, $R' = (\alpha A_1)^{1/2}$, and $r = |z| < 1$; A_b, D_b as in Lemma 1.

PROOF. Taking the logarithmic derivative of both sides of (1) we get

$$\begin{aligned}
\operatorname{Re}\left\{\frac{zP'(z)}{P(z)}\right\} &= (1-\beta)\operatorname{Re}\left\{\frac{zw'(z)}{(1-w(z))(1-\beta w(z))}\right\} \\
&\geqq (1-\beta)\left(\operatorname{Re}\left\{\frac{w(z)}{(1-w(z))(1-\beta w(z))}\right\}\right. \\
&\qquad\qquad \left. - \left|\frac{zw'(z) - w(z)}{(1-w(z))(1-\beta w(z))}\right|\right) \\
&\geqq (1-\beta)^{-1}\left(\operatorname{Re}\left\{P(z) + \frac{\beta}{P(z)} - (1+\beta)\right\}\right. \\
&\qquad\qquad \left. - \frac{|z|^2\,|P(z) - \beta|^2 - |P(z) - 1|^2}{(1 - |z|^2)\,|P(z)|}\right)
\end{aligned}$$

where the last inequality follows by virtue of Lemma 3 and the triangle inequality. The proof is concluded by applying Lemma 2 with $k=1$ and $\beta = 2\alpha - 1$.

Figure 1 shows the transitional curve determined by $R' = R_b$ for various values of r from 0.3 to 0.6.

REMARK 2. From Remark 1, $R_1 \leqq R_b$ for $b \in [0, 1]$. Thus $\alpha A_1 \leqq R_b^2$ whenever $\alpha A_1 \leqq R_1^2$ or equivalently, when

$$\begin{aligned}
0 &\leqq (1 + (2\alpha - 1)r^2)(1 - r) - \alpha(1 - (2\alpha - 1)r^2)(1 + r) \\
&= (1 - \alpha)W(r, \alpha)
\end{aligned}$$

where

$$\begin{aligned}
W(r, \alpha) &= 1 - 3r - 3(2\alpha - 1)r^2 + (2\alpha - 1)r^3 \geqq 1 - 3r - 3r^2 + r^3 \\
&= (1 - r)(1 - 4r + r^2) \geqq 0
\end{aligned}$$

for each $\alpha \in [0, 1)$. Hence (12) is valid for all $\alpha \in [0, 1)$, $b \in [0, 1]$ and $r \in [0, 2-3^{1/2}]$. Thus the transitional curve for $r=2-3^{1/2}$ would just touch the upper right-hand corner of the square in Figure 1.

3. **The class** $\mathscr{P}'_a(\alpha)$. Let $\mathscr{P}'_a(\alpha)$ denote the class of functions $F(z)=z+a(1-\alpha)z^2+\cdots$ such that $F'(z) \in \mathscr{P}_a(\alpha)$. Results concerning distortion and regions covered by the class together with other references may be found in **[1]**. A radius of convexity theorem is also presented in **[1]** for the class $\mathscr{P}'_a(\alpha)$; the result, however, is exact only for the case $\alpha=0$. Here we will produce a sharp estimate for all $a \in [0, 1]$ and $\alpha \in [0, 1)$.

From Nehari **[2]** it is known that $F(z)$ maps $|z|<r$ onto a convex region if $\operatorname{Re}\{1+zF''(z)/F'(z)\}>0$ for $|z|<r$.

LEMMA 4. *If $F(z) \in \mathscr{P}'_a(\alpha)$, then, for all $\alpha \in [0, 1)$ and $a \in [0, 1]$,*

$$\operatorname{Re}\left\{1+\frac{zF''(z)}{F'(z)}\right\}$$

$$\geqq \frac{1+4\alpha ar+(6\alpha-4+4a^2\alpha)r^2+4(2\alpha-1)ar^3+(2\alpha-1)r^4}{(1+2\alpha ar+(2\alpha-1)r^2)(1+2ar+r^2)} \quad \text{if } R_a \geqq R' \tag{14}$$

$$\geqq \frac{(1-2\alpha)-A_1+2(\alpha A_1)^{1/2}}{1-\alpha} \quad \text{if } R_a \leqq R' \tag{15}$$

where

$$R_a = \frac{1+2\alpha ar+(2\alpha-1)r^2}{1+2ar+r^2}, \qquad R' = (\alpha A_1)^{1/2}, \qquad r=|z|<1.$$

PROOF Let $P(z)=F'(z)$, then $\operatorname{Re}\{1+zF''(z)/F'(z)\}=1+\operatorname{Re}\{zP'(z)/P(z)\}$ and we may apply Theorem 1.

THEOREM 2. *Each $F(z) \in \mathscr{P}'_a(\alpha)$ maps $|z|<R$ onto a convex region where R is the smallest positive root of the equation*

$$1+4\alpha ar+(6\alpha-4+4a^2\alpha)r^2+4(2\alpha-1)ar^3+(2\alpha-1)r^4=0$$

if $R_a \geqq R'$, and $R=(1+(\alpha^{-1}-1)^{1/2})^{1/2}$ if $R_a \leqq R'$, R_a and R' are as in Lemma 4. The result is sharp for each $\alpha \in [0, 1)$ and $a \in [0, 1]$.

PROOF. Apply Lemma 4. For sharpness consider

$$F(z) = -(1-2\alpha)z+(1-\alpha)((1-a)\cdot\log(1+z)-(1+c)\cdot\log(1+z)) \quad \text{if } R_a \geqq R',$$

$$F(z) = -(1-2\alpha)z+(1-\alpha)((1-c)\cdot\log(1+z)-(1+c)\cdot\log(1-z)) \quad \text{if } R_a \leqq R'$$

where c is determined from $R_c=R'$.

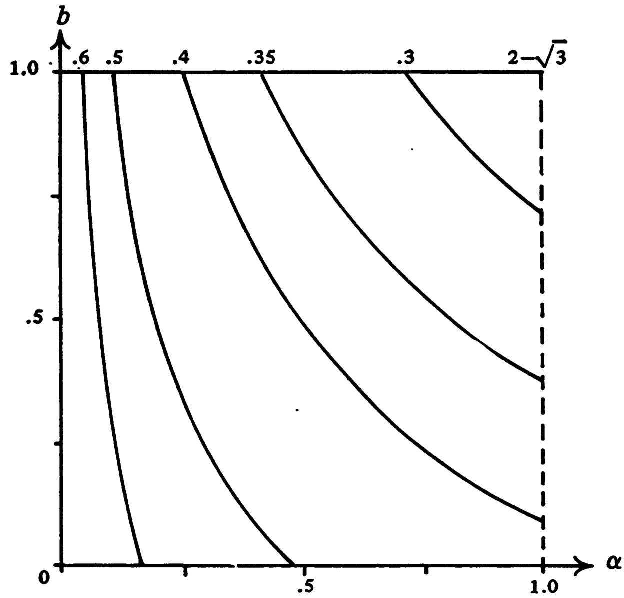

FIGURE 1

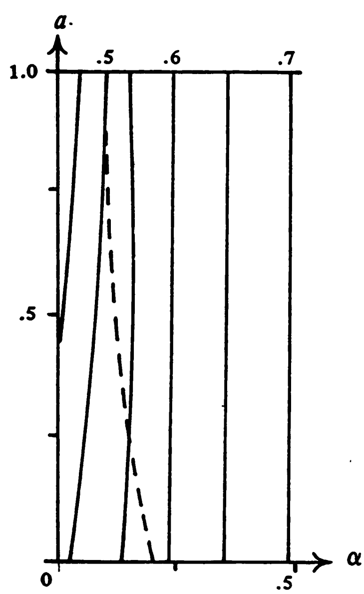

FIGURE 2

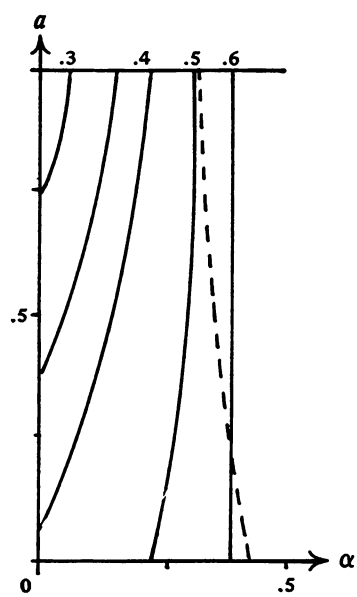

FIGURE 3

Figure 2 shows the transitional line $R_a = R'$ and displays some level curves for various values of the radius of convexity.

4. **The class** $\mathscr{S}_a^*(\alpha)$. Let $\mathscr{S}^*(\alpha)$ represent the class of functions $f(z)=z+a_2z^2+\cdots$ which are analytic and for which $\mathrm{Re}\{zf'(z)/f(z)\}>\alpha$ for $\alpha \in [0, 1)$ and $|z|<1$. A. Schild [3] has shown that $|a_2|\leqq 2(1-\alpha)$ and obtained results on the radius of convexity for a subclass of $\mathscr{S}^*(\alpha)$. Singh and Goel [4] obtained the exact radius of convexity for the entire class $\mathscr{S}^*(\alpha)$. Define

$$\mathscr{S}_a^*(\alpha) = \{f(z) = z + 2a(1-\alpha)z^2 + \cdots : f(z) \in \mathscr{S}^*(\alpha)\}.$$

D. Tepper [5] found the exact radius of convexity for $\mathscr{S}_a^*(0)$. Here we determine a sharp estimate for the entire class $\mathscr{S}_a^*(\alpha)$.

Lemma 5. *If $f(z) \in \mathscr{S}_a^*(\alpha)$, then, for all $\alpha \in [0, 1)$ and $a \in [0, 1]$,*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}$$

$$\geqq \frac{1+(6\alpha-2)ar+(4\alpha^2a^2+8\alpha-6)r^2+(8\alpha^2-2\alpha-2)ar^3+(2\alpha-1)^2r^4}{(1+2ar+r^2)(1+2\alpha ar+(2\alpha-1)r^2)} \tag{16}$$

if $R_a \geqq R'$

$$\geqq \frac{2(\alpha(2-\alpha)A_1)^{1/2} - \alpha - A_1}{1-\alpha} \quad \text{if } R_a \leqq R' \tag{17}$$

where

$$R_a = \frac{1 + 2\alpha ar + (2\alpha-1)r^2}{1 + 2ar + r^2}, \quad R' = \left(\frac{\alpha}{2-\alpha} A_1\right)^{1/2}, \quad r = |z| < 1.$$

Proof. Let $P(z)=zf'(z)/f(z)$, then $\mathrm{Re}\{1+zf''(z)/f'(z)\}=\mathrm{Re}\{P(z)\}+\mathrm{Re}\{zP'(z)/P(z)\}$. Now apply Lemma 2 with $\beta=2\alpha-1$ and $k=3-2\alpha$.

Theorem 3. *Each $f(z) \in \mathscr{S}_a^*(\alpha)$ maps $|z|<R$ onto a convex region where R is the smallest positive root of the equation*

$$1 + (6\alpha-2)ar + (4\alpha^2a^2 + 8\alpha - 6)r^2 + (8\alpha^2 - 2\alpha - 2)ar^3 + (2\alpha-1)^2r^4 = 0$$

if $R_a \geqq R'$ and $R=((5\alpha-1)/(4\alpha^2-\alpha+1+4\alpha(\alpha^2-3\alpha+2)^{1/2}))^{1/2}$ if $R_a \leqq R'$ where R_a and R' are as in Lemma 5. The result is sharp for each $\alpha \in [0, 1)$ and $a \in [0, 1]$.

Proof. Apply Lemma 5. For sharpness, let

$$f(z) = z/(1 - 2az + z^2)^{1-\alpha}, \quad \text{if } R_a \geqq R',$$

and

$$f(z) = z/(1 - 2cz + z^2)^{1-\alpha}, \quad \text{if } R_a \leqq R',$$

where c is determined from $R_c = R'$.

Figure 3 gives some level curves and the transitional curve for the class $\mathscr{S}_a^*(\alpha)$.

REMARK 3. By setting $a=0$ in Theorems 2 and 3 we could obtain sharp results on odd functions in the two classes $\mathscr{P}'(\alpha)$ and $\mathscr{S}^*(\alpha)$ for $\alpha \in [0, 1)$.

REFERENCES

1. C. P. McCarty, *Functions with real part greater than* α, Proc. Amer. Math. Soc. **35** (1972), 211–216.

2. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

3. A. Schild, *On starlike functions of order* α, Amer. J. Math. **87** (1965), 65–70. MR **30** #4929.

4. V. Singh and R. M. Goel, *On radii of convexity and starlikeness of some classes of functions*, J. Math. Soc. Japan **23** (1971), 323–339. MR **43** #7617.

5. D. E. Tepper, *On the radius of convexity and boundary distortion of schlicht functions*, Trans. Amer. Math. Soc. **150** (1970), 519–528. MR **42** #3268.

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