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AREA OF BERNSTEIN-TYPE POLYNOMIALS

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ABSTRACT. Bernstein polynomials in one variable are known to be total-variation diminishing when compared to the approximated function f. Here we consider the two variable case and give a counterexample to show they are not area-diminishing. Sufficient conditions are then given on a continuous function f to insure convergence in area. A similar theorem is proved for Kantorovitch polynomials in the case f is summable.

We consider the two-dimensional Bernstein polynomials $B_{n,m}f$, and the corresponding Kantorovitch polynomials $K_{n,m}f$, for functions $z=f(x, y)$ defined on the unit square Q. Sufficient conditions are given to insure the convergence in area of these polynomials. In particular if f is summable and generalized absolutely continuous on Q, then $LK_{n,m}f \to \Phi f$ where L is Lebesgue area, and Φ is the Cesari-Goffman generalized area; if f is continuous and ACT, with R-integrable Tonelli lengths, then $LB_{n,m}f \to Lf$.

For any f defined on all of Q,

$$B_{n,m}f(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{m} f\left(\frac{r}{n}, \frac{s}{m}\right) p_{n,r}(x) p_{m,s}(y)$$

where $p_{N,R}(t) = \binom{N}{R} t^R (1-t)^{N-R}$.

For summable f on Q,

$$K_{n,m}f(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{m} I_{r,s} p_{n,r}(x) p_{m,s}(y)$$

where

$$I_{r,s} = (n+1)(m+1) \int_{r/(n+1)}^{(r+1)/(n+1)} \int_{s/(m+1)}^{(s+1)/(m+1)} f(\xi, \eta)\, d\xi\, d\eta.$$

If f is continuous, $B_{n,m}f$ and $K_{n,m}f$ converge uniformly to f. Although the behavior of $B_{n,m}f$ for discontinuous functions is quite erratic,

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e.g. [L, p. 28], and [P_1], we have

PROPOSITION 1. *If f is summable on Q, $K_{n,m}f$ converges in the L_1 sense to f.*

PROOF. For all m, n, $\int_0^1 \int_0^1 K_{n,m}f = \int_0^1 \int_0^1 f$ because $\int_0^1 p_{N,R}(t)\,dt = 1/(N+1)$ for any N and $R=0, 1, \cdots, N$. Hence $\|K_{n,m}f\|_1 \leqq \|f\|_1$. Choose a continuous h such that $\|f-h\|_1 \leqq \varepsilon/3$. Then

$$\begin{aligned}\|f - K_{n,m}f\|_1 &\leqq \|f - h\|_1 + \|h - K_{n,m}h\|_1 + \|K_{n,m}h - K_{n,m}f\|_1 \\ &\leqq 2\|f - h\|_1 + \|h - K_{n,m}h\|_1.\end{aligned}$$

Since h is continuous, the last term is also at most $\varepsilon/3$ for large m and n, which completes the proof.

Cesari and later Goffman have defined equivalent areas for summable functions on Q. We give Goffman's version [G_1]. Let

$$\Phi f \equiv \inf_{\{p_i\}} \liminf_{i\to\infty} L(p_i)$$

where p_i are quasilinear functions converging L_1 to f and the inf is taken over all such sequences of p_i. Φ is lower semicontinuous with respect to L_1 convergence and coincides with L for continuous f.

If $f(x, y)$ is continuous, the linear variation for fixed y is denoted by $V_x f(y)$; similarly $V_y f(x)$. Their Lebesgue integrals, the Tonelli variations are $V_x f = \int_0^1 V_x f(y)\,dy$ and $V_y f = \int_0^1 V_y f(x)\,dx$. Correspondingly for summable $f(x, y)$, the linear generalized variations are $\varphi_x f(y)$ and $\varphi_y f(x)$ where variation in each case is computed only over points of linear approximate continuity. The generalized Tonelli variations are $\varphi_x f = \int_0^1 \varphi_x(y)\,dy$ and $\varphi_y f = \int_0^1 \varphi_y(x)\,dx$. For continuous f and g,

$$L(f + g) \leqq Lf + V_x g + V_y g \tag{1a}$$

and for summable f and g,

$$\Phi(f + g) \leqq \Phi f + \varphi_x g + \varphi_y g. \tag{1b}$$

A continuous $f(x, y)$ is ACT if $V_x f$ and $V_y f$ are finite and f is absolutely continuous on almost all lines parallel to each coordinate axis. A summable f is said to be gACT if $\varphi_x f$ and $\varphi_y f$ are finite, and there exists an $h \sim g$ such that h is absolutely continuous on almost all lines parallel to each coordinate axis. Functions of gACT type may be "essentially discontinuous" i.e. every $h \sim f$ is nowhere continuous [G_2].

For finite valued $f(x)$ on $[0, 1]$,

$$B_n f(x) \equiv \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{n,r}(x)$$

and for summable f,

$$K_n f(x) \equiv \sum_{r=0}^{n} (n+1)\left(\int_{r/(n+1)}^{(r+1)/(n+1)} f(\xi)\, d\xi\right) p_{n,r}(x).$$

Let V be total variation, φ be variation over points of approximate continuity, l the Jordan length, and λ the length over points of approximate continuity. Then for all n,

$$\text{(2)} \qquad \begin{array}{ll} \text{(a) } VB_n f \leqq Vf, & \text{(c) } lB_n f \leqq lf, \\ \text{(b) } VK_n f \leqq \varphi f, & \text{(d) } \lambda K_n f \leqq \lambda f. \end{array}$$

Part (a) is in **[L]**; (b) is in **[P$_2$]**; (c) and (d) follow from (a) and (b) by an integral-geometric formula of Cauchy and Steinhaus **[P$_2$]**. In virtue of the lower semicontinuity of V and l with respect to uniform convergence, and of φ and λ with respect to L_1 convergence, all four functionals converge as $n \to \infty$. It is thus reasonable to conjecture $LB_{n,m}f \to Lf$ and $LK_{n,m}f \to \Phi f$ as $n, m \to \infty$ for appropriate classes of functions.

There is a major difference in the two variable case however. Construct a C^∞ "rounded spike" function f_ε on Q which vanishes off a circular neighborhood C_ε of $(\frac{1}{2}, \frac{1}{2})$ and assumes the value 1 at $(\frac{1}{2}, \frac{1}{2})$. By making the spike sufficiently thin, $Lf_\varepsilon = 1 + \varepsilon$ for arbitrarily small positive ε. On the other hand $B_{22} f_\varepsilon = 4xy(1-x)(1-y)$ and is independent of the base radius r_ε of the spike. Hence, though f_ε is C^∞, $LB_{22} f_\varepsilon > 1 + \varepsilon = Lf_\varepsilon$ for some ε in contrast to the relations (2). We now state the theorems.

THEOREM 1. *If f is gACT, then* $\lim_{n,m\to\infty} LK_{m,n} f = \Phi f$.

PROOF. Φ is lower-semicontinuous with respect to L_1 convergence, so by Proposition 1, $\liminf_{n,m\to\infty} LK_{n,m} f \geqq \Phi f$.

By (1b),

$$\begin{aligned} \Phi f &\leqq \liminf LK_{n,m} f \leqq \limsup LK_{n,m} f = \limsup \Phi K_{n,m} f \\ &\leqq \Phi f + \limsup \varphi_x(K_{n,m} f - f) + \limsup \varphi_y(K_{n,m} f - f). \end{aligned}$$

It will be sufficient then to show (say) $\varphi_x(K_{n,m}f - f) \to 0$. Since f is gACT, $\partial f/\partial x$ is summable, where $\partial f/\partial x$ is the partial derivative with sets of measure zero neglected in the difference quotient **[G$_1$]**. Pick h continuously differentiable on Q such that $\|(\partial f/\partial x) - h\|_1 < \varepsilon/3$; i.e. $\varphi_x(f - H) < \varepsilon/3$ where $H(x, y) = \int_0^x h(t, y)\, dt$. Thus

$$\varphi_x(K_{n,m} f - f) \leqq \varphi_x(f - H) + V_x(H - K_{n,m} H) + V_x(K_{n,m} H - K_{n,m} f).$$

The first term is $< \varepsilon/3$, and so is the second for large n and m because $(\partial K_{n,m} H/\partial x) \to (\partial H/\partial x)$, since H is C^1. The proof of this follows from showing $|(\partial K_{n,m}/\partial x) - (\partial B_{n,m}/\partial x)|$ to be small, and then using the corresponding result for $B_{n,m}$ which is proved in **[B]**.

For the third term, we need a lemma which holds for any summable function.

LEMMA. *For $F(x, y)$ summable on Q and all m and n, $V_xK_{n,m}F \leqq \varphi_x F$ (and $V_yK_{n,m}F \leqq \varphi_y F$).*

PROOF.

$$\begin{aligned} V_xK_{n,m}F &= \int_0^1\int_0^1 \left|\frac{\partial K_{n,m}F}{\partial x}\right| dx\,dy \\ &= n\int_0^1\int_0^1 \left|\sum_{s=0}^{m}\sum_{r=0}^{n-1}(I_{r+1,s} - I_{r,s})p_{n-1,r}(x)p_{m,s}(y)\right| dx\,dy \\ &\leqq n\sum_{r=0}^{n-1}\sum_{s=0}^{m}\int_0^1\int_0^1 |I_{r+1,s} - I_{r,s}|\, p_{n-1,r}(x)p_{m,s}(y)\,dx\,dy \\ &= \frac{1}{m+1}\sum_{r=0}^{n-1}\sum_{s=0}^{m}|I_{r+1,s} - I_{r,s}|. \end{aligned}$$

But

$$\begin{aligned} |I_{r+1,s} - I_{r,s}| \leqq (m+1)\int_{s/(m+1)}^{(s+1)/(m+1)} &(n+1) \\ &\cdot\left|\int_{(r+1)/(n+1)}^{(r+2)/(n+1)} F(\xi,\eta)\,d\xi - \int_{r/(n+1)}^{(r+1)/(n+1)} F(\xi,\eta)\,d\xi\right| d\eta \end{aligned}$$

and so

$$\begin{aligned} V_xK_{n,m}F \leqq \int_0^1 (n+1)\sum_{r=0}^{n-1}\Bigg|&\int_{(r+1)/(n+1)}^{(r+2)/(n+1)} F(\xi,\eta)\,d\xi \\ &- \int_{r/(n+1)}^{(r+1)/(n+1)} F(\xi,\eta)\,d\xi\Bigg|\,d\eta \end{aligned} \tag{3}$$

For almost all $\eta \in [0, 1]$, $F(\xi, \eta)$ is a summable function of ξ. For these η, the expression inside the first integral is at most $\varphi_x F(\eta)$. The proof is essentially that of (2)(b). Thus the right hand side of (3) is at most $\int_0^1 \varphi_x F(\eta)\,d\eta = \varphi_x F$ which completes the proof.

Now let $F = H - f$. F is summable, and so by the lemma

$$V_x(K_{n,m}H - K_{n,m}f) = V_x(K_{n,m}(H-f)) \leqq \varphi_x(H-f) < \varepsilon/3.$$

Hence $\varphi_x(K_{n,m}f - f) < \varepsilon$ for large n and m which completes the proof of Theorem 1.

For the next theorem, set $l_x f = \int_0^1 l_x f(y)\,dy$ where $l_x f(y)$ is the Jordan length in the x-direction of a section at y. Similarly define $l_y f$.

THEOREM 2. *If f is ACT and $l_x f$ and $l_y f$ are R-integrable, then $\lim_{n,m\to\infty} LB_{n,m}f = Lf$.*

PROOF. Since $B_{n,m}f \to f$ uniformly, $\liminf_{n,m\to\infty} LB_{n,m}f \geqq Lf$. By (1a), it is sufficient to show as in Theorem 1, that (say) $V_x(B_{n,m}f-f)\to 0$. Let h and H be as in Theorem 1 with $V_x(f-H)<\varepsilon/4$. Then

$$V_x(B_{n,m}f-f) \leqq V_x(f-H)+V_x(H-B_{n,m}H)+V_x(B_{n,m}(H-f)).$$

The first term is at most $\varepsilon/4$, as is the second for large n and m, because $(\partial B_{n,m}H/\partial x)\to(\partial H/\partial x)$ uniformly [B]. For the third term, it is necessary to show $V_x(f-H)(y)$ is R-integrable.

Since $l_x f$ is R-integrable, $l_x f(y)$ and hence $V_x f(y)$ is bounded for $y \in [0, 1]$. Since H is C^1, $V_x(f-H)(y)$ is bounded. In addition, $V_x(f-H)(y)$ is continuous almost everywhere. To see this, pick y_0 from the full measure set where simultaneously $f(x, y_0)$ is absolutely continuous as a function of x, and $l_x f(y)$ is continuous as a function of y. Consider a sequence $y_n\to y_0$, and correspondingly the $l_x f(y_n)$ and $l_x H(y_n)$. Since H is C^1, $H(x, y_0)$ is an absolutely continuous function of x. By theorems in [A-L], $l_x(f-H)(y_n)\to l_x(f-H)(y_0)$ which implies $V_x(f-H)(y_n)\to V_x(f-H)(y_0)$. Thus $V_x(f-H)(y)$ is continuous at almost all y and is R-integrable.

For arbitrary $F(x, y)$, a computation similar to the lemma shows

$$V_xB_{n,m}F \leqq \frac{1}{m+1}\sum_{s=0}^{m} V_xF\left(\frac{s}{m}\right)$$

for all n, m. Thus

$$V_xB_{n,m}(H-f) \leqq \frac{1}{m+1}\sum_{s=0}^{m} V_x(H-f)\left(\frac{s}{m}\right) \tag{4}$$

which converges to $V_x(H-f)$ by R-integrability of $V_x(H-f)(y)$. Hence for large m and all n, the right hand side of (4) is less than $2(\varepsilon/4)=\varepsilon/2$. For the same m and n,

$$V_x(B_{n,m}f-f) \leqq \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon,$$

and the same computation for y shows $V_y(B_{n,m}f-f)\to 0$. Therefore $\limsup LB_{n,m}f \leqq Lf$ which completes the proof.

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