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THE PROBLEM OF EIGENVALUES IN SOME SINGULAR HOMOGENEOUS VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. It is shown that when the kernel of a homogeneous Volterra integral equation is singular, it is possible for there to be a continuous spectrum of eigenvalues.

1. **Introduction.** Consider the Volterra integral equation

$$\phi(x) = f(x) + \lambda \int_0^x K(x, y)\phi(y)\, dy. \tag{1}$$

It is well known [1] that, if K is continuous or weakly singular, the power series expression in λ for the resolvent is convergent for all λ, and that, consequently, the homogeneous Volterra equation

$$\phi(x) = \lambda \int_0^x K(x, y)\phi(y)\, dy \tag{2}$$

does not possess any eigenvalues. What does not seem to be well known, [2] however, is that there can be, under certain conditions, a continuous eigenvalue spectrum when the kernel is singular.

This can be shown by a very simple example. From the result

$$x^n = n \int_0^x y^{n-1}\, dy \qquad (\mathrm{Re}(n) > 0) \tag{3}$$

it follows that a solution of the Volterra type integral equation

$$\phi(x) = \lambda \int_0^x x^{-1} \exp\{x - y\}\phi(y)\, dy \tag{4}$$

with the singular kernel $x^{-1} \exp\{x-y\}$ is

$$\phi_n(x) = \exp(x)x^{n-1} \qquad (\mathrm{Re}(n) > 0) \tag{5}$$

which has associated with it the eigenvalue n. There is thus a continuous spectrum of eigenvalues for the integral equation (4). An alternative

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representation of (4) would be given by writing $x\phi(x)=\psi(x)$, giving

$$\psi(x) = \lambda \int_0^x y^{-1} \exp\{x - y\}\psi(y)\, dy. \tag{6}$$

In this case the kernel $y^{-1}\exp\{x-y\}$ is still singular.

2. **Analysis.** Consider now the Volterra type integral equation

$$a(x)\phi(x) = \lambda \int_0^x m(x, y)\phi(y)\, dy \tag{7}$$

the kernel of which is $\{a(x)\}^{-1}m(x, y)$. Suppose now that $m(x, y)$ is of the form $m(x-y)$. This will simplify the analysis somewhat, but does not affect the ideas involved. Suppose that

$$a(x) = \sum_{s=0}^{\infty} a_s x^{s+\alpha}, \tag{8}$$

$$m(x) = \sum_{s=0}^{\infty} m_s x^{s+\mu}, \tag{9}$$

and look for a solution of the form

$$\phi(x) = \sum_{s=0}^{\infty} \phi_s x^{s+\xi}, \tag{10}$$

α and μ are known, ξ is to be determined. Obviously a_0, m_0, ϕ_0 are nonzero.

Equation (7) can now be rewritten in the form

$$\sum_{s=0}^{\infty} a_s x^{s+\alpha} \sum_{s=0}^{\infty} \phi_t x^{t+\xi} = \lambda \int_0^x \sum_{s=0}^{\infty} m_s (x - y)^{s+\mu} \sum_{t=0}^{\infty} \phi_t x^{t+\xi}\, dx$$

which simplifies further to

$$\begin{aligned} \sum_{s=0}^{\infty} x^{s+t+\alpha+\xi} \sum_{t=0}^{s} a_{s-t}\phi_t & \\ &= \lambda \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} x^{s+t+\mu+\xi+1} m_s \phi_t \int_0^1 (1 - Z)^{s+\mu} Z^{t+\xi}\, dZ \\ &= \lambda \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} x^{s+t+\mu+\xi+1} m_s \phi_t B(s + \mu + 1, t + \xi + 1) \\ &= \lambda \sum_{s=0}^{\infty} x^{s+\mu+\xi+1} \sum_{t=0}^{s} m_{s-t}\phi_t B(s - t + \mu + 1, t + \xi + 1). \end{aligned} \tag{11}$$

It will be noted that for the integrals to converge, it is necessary that $\mathrm{Re}(\mu+1)$ and $\mathrm{Re}(\xi+1)$ should be positive. In order that the leading term of both series be the same, it follows that $\alpha+\xi=\mu+\xi+1$, which implies that $\alpha=\mu+1$. (Although apparently a possible solution would be given by

$\alpha-\mu-1$ being equal to an integer N, it can easily be seen that this would imply ϕ_s vanishing for $s<N$ and so is irrelevant.) Equation (7) now assumes the form

$$\sum_{s=0}^{\infty} x^s \sum_{t=0}^{s} a_{s-t}\phi_t = \lambda \sum_{s=0}^{\infty} x^s \sum_{t=0}^{s} m_{s-t}\phi_t B(s-t+\mu+1, t+\xi+1) \tag{12}$$

as $x^{\alpha+\xi}$ cancels. It follows from equation (12) that

$$\sum_{t=0}^{s} (a_{s-t} - \lambda m_{s-t} B(s-t+\mu+1, t+\xi+1))\phi_t = 0, \qquad s \geqq 0. \tag{13}$$

The eigenvalue is determined by taking the case $s=0$. This gives

$$a_0 - \lambda m_0 B(\mu+1, \xi+1) = 0 \tag{14}$$

or

$$\lambda = a_0/\{m_0 B(\mu+1, \xi+1)\}. \tag{15}$$

Now ξ is not defined, save by the convergency condition referred to previously, and so it follows that the spectrum of λ as given by equation (15) is continuous. The set of equations (13) can, using (15), be rewritten as

$$\sum_{t=0}^{s} \left[a_{s-t} - \frac{m_{s-t} a_0 B(s-t+\mu+1, t+\xi+1)}{m_0 B(\mu+1, \xi+1)} \right] \phi_t = 0. \tag{16}$$

The set of equations (16) gives recurrence relations for ϕ_s in terms of $\phi_{s-1}, \cdots, \phi_0$. (It can easily be verified that the coefficient of ϕ_n does not vanish in (16) when $s=n$.) It will be noted that the ϕ_s are in fact functions of ξ. Thus the sum of the series

$$\sum_{s=0}^{\infty} \phi_s(\xi) x^{s+\xi}, \tag{17}$$

where the $\phi_s(\xi)$ are defined by equation (16) is, if it converges, and if $\mathrm{Re}(\xi+1)$ is positive, an eigenfunction for the integral equation (7), the corresponding eigenvalue being given by equation (15).

References

1. V. I. Smirnov, *Course in higher mathematics*. Vol. IV, 3rd ed., GITTL, Moscow, 1953; English transl., Pergamon Press, Oxford; Addison-Wesley, Reading, Mass., 1964, p. 136. MR **31** #1333.

2. The author has consulted 17 books on integral equations, but has found no reference to this.

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