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ABSOLUTE CONTINUITY OF EIGENVECTORS OF TIME-VARYING OPERATORS

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ABSTRACT. If $K(t)$ is a compact, selfadjoint operator function of a real variable t with distinct eigenvalues at each t, we show that the eigenvalues and eigenvectors are absolutely continuous and that $\{K(t)\}$ is a commuting set provided that $K(t)$ commutes with its time derivative $K'(t)$ at each t. The distinct eigenvalue condition is shown to be necessary.

1. **Introduction.** Let H be a real Hilbert space, i.e. the inner product on H is real valued. Let $K(t):H\to H$ be a compact linear operator for each t in a closed real interval J, and assume $K(t)$ is absolutely continuous in the operator norm on J. We will prove the following:

THEOREM 1. *Assume $\lambda(t)$ is a distinct (at each $t \in J$), absolutely continuous eigenvalue function of $K(t)$ on J, and let $\varphi(t)$ be a corresponding normalized eigenvector function. Then $\varphi(t)$ can be chosen absolutely continuous on J.*

This result can be used inductively in connection with the usual spectral theory for compact, selfadjoint operators to prove the following:

THEOREM 2. *Assume $K(t)$ is selfadjoint and has only distinct, nonzero eigenvalues at each $t \in J$. Then there is a sequence of absolutely continuous (on J) eigenvalue and (corresponding, normalized) eigenvector functions: $\{\lambda_n(t), \varphi_n(t)\}_n$ which include the spectrum of $K(t)$ at each t, and*

$$K(t)x = \sum \lambda_n(t)\varphi_n(t)\langle\varphi_n(t), x\rangle \qquad (x \in H \text{ and } t \in J).$$

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[1] This problem occurred in connection with the problem of factoring unbounded operator functions during my doctoral research at Boston University, 1971. I wish to thank Professor Marvin I. Freedman for helpful suggestions.

The results concerning continuity of eigenvalues and eigenvectors have a natural application to perturbation methods. Also Theorem 2 may be used to prove:

THEOREM 3. *Let $K(t)$ be selfadjoint and have only distinct, nonzero eigenvalues in J. Assume $K(t)$ commutes with its derivative $K'(t)$ almost everywhere in J. Then $\{K(t)|t \in J\}$ is a commuting set.*

All the theorems can be extended to corresponding theorems for unbounded operator functions which have a compact inverse for each $t \in J$. Theorems similar to 2 and 3 are proven for a square matrix function $K(t)$ which is not selfadjoint (and can have the eigenvalue 0) in A. Acker [**1**]. An example in §4 shows the distinct eigenvalue condition is necessary in each above theorem.

2. **The proof of Theorem 1.** The proof is simplified by agreeing to call an operator A "δ-positive" (for a specified $\delta>0$) if $|Ax|\geqq\delta|x|$ for all vectors x orthogonal to the nullspace of A. One can check that any operator of the form: $K-\lambda I$, where K is compact, I is the identity, and $\lambda\neq 0$, is δ-positive for some, possibly very small, positive δ. If we set $A(t)=K(t)-\lambda(t)I$, then Theorem 1 can be restated as follows:

LEMMA. *For each $t \in J$, let the operator $A(t):H\to H$ be $\delta(t)$-positive (where $\delta(t)>0$) and have 0 as a distinct eigenvalue. Assume the operator function $A(t)$ is absolutely continuous on J in the operator norm. Then the corresponding (to 0) normalized eigenvector function $\varphi(t)$ can be chosen absolutely continuous on J.*

PROOF. At each t, $\varphi(t)$ is unique only to a factor of ± 1. (Complex multiples are not in H.) For a specified choice of $\varphi(t)$ and t_0, $t \in J$, let $p(t_0, t)=\langle\varphi(t_0), \varphi(t)\rangle\varphi(t_0)$ and $p_\perp(t_0, t)=\varphi(t)-p(t_0, t)$ so that $p_\perp(t_0, t)$ is orthogonal to $\varphi(t_0)$ and $|p(t_0, t)|^2+|p_\perp(t_0, t)|^2=1$. One can check that

$$A(t_0)p_\perp(t_0, t) = [A(t_0)-A(t)]\varphi(t). \tag{1}$$

Since $A(t_0)$ is $\delta(t_0)$-positive and $|\varphi(t)|=1$ for all t, equation 1 implies (where $\|\cdot\|$ is the operator norm) that

$$\delta(t_0)\,|p_\perp(t_0, t)| \leqq \|A(t) - A(t_0)\| \quad \text{for all } t. \tag{2}$$

We conclude that $|p_\perp(t_0, t)|\to 0$, $|p(t_0, t)|\to 1$, and $|\langle\varphi(t_0), \varphi(t)\rangle|\to 1$ as $t\to t_0$. For any $t_0 \in J$, there is a relatively open interval $I(t_0)$ containing t_0 on which $\langle\varphi(t_0), \varphi(t)\rangle\neq 0$. Redefine $\varphi(t)$ at each $t \in I(t_0)$ by multiplication by ± 1 so that $\langle\varphi(t_0), \varphi(t)\rangle$ is positive. Then $\langle\varphi(t_0), \varphi(t)\rangle\to 1$ and $\varphi(t)\to\varphi(t_0)$ as $t\to t_0$. Let $I^*(t_0)$ be a relatively open subinterval of $I(t_0)$ in which $\langle\varphi(t_0), \varphi(t)\rangle>\frac{1}{2}$ and $|\varphi(t_0)-\varphi(t)|<\frac{1}{2}$. It is easily seen that the product $\langle\varphi(t_1), \varphi(t_2)\rangle$ is positive whenever t_1 and t_2 are in $I^*(t_0)$, and therefore the preceding argument shows that the redefined function $\varphi(t)$ is continuous throughout $I^*(t_0)$.

It is intuitive that if $A(t)$ and $\varphi(t)$ (which spans the nullspace) are continuous, then $A(t)$ is uniformly δ-positive on a sufficiently small interval $I^{**}(t_0)$ about t_0. This is proven starting with the equation:

$$A(t)\psi = [A(t) - A(t_0)]\psi + A(t_0)[\psi - \langle\psi, \varphi(t_0) - \varphi(t)\rangle\varphi(t_0)]$$

which holds whenever ψ is orthogonal to $\varphi(t)$.

One can show that $|\varphi(t_1)-\varphi(t_2)|<\sqrt{2}|p_\perp(t_1, t_2)|$ for $t_1; t_2 \in I^*(t_0)$. Use this in connection with equation 2 and the uniform δ-positivity of $A(t)$ to obtain (for a fixed positive δ)

$$\delta\,|\varphi(t_1) - \varphi(t_2)| < \sqrt{2}\,\|A(t_1) - A(t_2)\| \quad (\text{whenever } t_1, t_2 \in I^{**}(t_0)).$$

This immediately shows $\varphi(t)$ is absolutely continuous on $I^{**}(t_0)$. The result is easily extended to the (compact) interval J.

3. **The proof of Theorem 3.** It will be shown that if $K(t)$ commutes with $K'(t)$ (written: $K(t)\sim K'(t)$) a.e. in J, then the absolutely continuous eigenvector functions $\varphi_n(t)$ of $K(t)$ are all constant. Assume $\varphi(t)$ is the eigenvector function corresponding to the eigenvalue function $\lambda(t)$ and let $A(t)=K(t)-\lambda(t)I$. Then $A'(t)$ exists a.e. and $A(t)\sim A'(t)$ a.e. in J. Since 0 is a distinct eigenvalue of $A(t)$, we conclude that $\varphi(t)$ is also an eigenvector of $A'(t)$, i.e. $A'(t)\varphi(t)=\alpha(t)\varphi(t)$ for a real function $\alpha(t)$. From $A(t)\varphi(t)=0$ we obtain $A'(t)\varphi(t)+A(t)\varphi'(t)=0$. Therefore $A(t)\varphi'(t)+\alpha(t)\varphi(t)=0$ a.e. The vector product of this equation with $\varphi(t)$ shows that $\alpha(t)=0$ a.e. in J. Therefore $\varphi(t)$ and $\varphi'(t)$ are both eigenvectors of $A(t)$ at the distinct eigenvalue 0, and it follows that: $\varphi'(t)=c(t)\varphi(t)$ for a real function $c(t)$. We find that $c(t)=\langle\varphi(t), \varphi'(t)\rangle$, so $c(t)$ is integrable, and, for $t_0 \in J$, the differential equation has a unique absolutely continuous solution $\varphi(t)=C(t)\varphi(t_0)$, where $C(t)=\int_{t_0}^{t} c(t')\,dt'$. $C(t)$ is real and continuous, and $|C(t)|=1$. Therefore $C(t)=1$ and $\varphi(t)=\varphi(t_0)$ for $t \in J$.

4. **The distinct eigenvalue condition is necessary.** This is shown by an example. Let K_+ and K_- be two square matrices, each with distinct eigenvalues, which do not commute. Define $K(t)$ as follows:

$$\begin{aligned} K(t) &= I + t^2K_- \quad \text{when } t \leqq 0, \\ &= I + t^2K_+ \quad \text{when } t \geqq 0. \end{aligned}$$

Then $K(t)$ has the following properties: 1. $K(t)$ and its eigenvalues are absolutely continuous on any finite interval, and the eigenvalues are distinct except at $t=0$. 2. $K'(t)$ exists and $K(t)\sim K'(t)$ at all t including 0. 3. The (continuous) eigenvectors of $K(t)$ are constant on $(-\infty, 0)$ and on $(0, \infty)$ and are simply the eigenvectors, respectively, of K_- and K_+. Thus they are not continuous across 0. 4. If $t_1<0<t_2$, then $K(t_1)$ and $K(t_2)$ do not commute.

REFERENCE

1. A. Acker, *Stability results for linear systems involving a time varying unbounded operator*, Doctoral Dissertation, Boston University, 1972, Appendix B.

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