PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

THE M.H.D. VERSION OF A THEOREM OF H. WEYL

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ABSTRACT. In his discussion of shock waves in arbitrary fluids, H. Weyl proves a theorem concerning the behavior of the entropy function along the Hugoniot curve. The analogous result is proven for the M.H.D. case.

The theorem of the title concerns the behavior of a function on a curve: The function represents entropy, the curve is the Hugoniot curve. Such a curve is determined for each point (V_0, p_0) in the positive quadrant of the (specific) volume-pressure plane. Weyl [3] shows for gas dynamics that the entropy function restricted to this curve has at most one critical point, and that (V_0, p_0) is the candidate. The significance of the result is that the entropy behaves in the physically expected way across shocks. For the general context the reader is referred to [2]; here we only want to point out how Weyl's clever argument works in the magnetohydrodynamic (m.h.d.) case.

The Hugoniot curve in the (V, p) plane corresponding to the point (V_0, p_0) is defined [1] by the equation

$$H(V,p) = e - e_0 + \tfrac{1}{2}(p + p_0)(V - V_0) + \phi(V) = 0. \tag{1}$$

Here $e=e(V,p)$ represents internal energy and $e_0=e(V_0,p_0)$. The function $\phi(V)$ is identically zero in the case of gas dynamics while in the m.h.d. case ϕ takes the form

$$\phi(V) = K(V - V_0)^3(V - V_1)^{-2} \tag{2}$$

where K and V_1 $(\neq V_0)$ are positive constants (see [1]).

We use the following facts and hypothesis:

(3) *$de=-p\,dV+T\,ds$, where T and s denote respectively the temperature and entropy.*

(4) *Given V and s, a unique value $p=p(V,s)$ is determined. For the function so defined, $\partial^2p/\partial V^2$ and $\partial p/\partial s$ are both positive.*

$$\phi(V_0) = \phi'(V_0) = 0 \quad \textit{and} \quad \phi''(V)(V - V_0) \geqq 0. \tag{5}$$

Received by the editors April 3, 1973.
AMS (MOS) subject classifications (1970). Primary 76L05, 76W05.

For the ϕ of m.h.d. this is satisfied. In particular we compute

$$\phi'(V) = 3K(V - V_0)^2(V - V_1)^{-2} - 2K(V - V_0)^3(V - V_1)^{-3},$$
$$\phi''(V) = 6K(V - V_0)(V - V_1)^{-2} - 12K(V - V_0)^2(V - V_1)^{-3} + 6K(V - V_0)^3(V - V_1)^{-4},$$
$$\phi''(V)(V - V_0) = 6K\{(V - V_0)(V - V_1)^{-1} - (V - V_0)^2(V - V_1)^{-2}\}^2 \geqq 0.$$

The main lemma (8) used to examine the behavior of s on the set where $H=0$ concerns the behavior of s on a different family of curves. These are defined to be those curves restricted to which the one-form $\omega = dH - T\,ds$ vanishes.

From (1), using (3), we compute

$$dH = [\tfrac{1}{2}(p_0 - p) + \phi'(V)]\,dV + \tfrac{1}{2}[V - V_0]\,dp + T\,ds.$$

Thus the integral curves of $\omega=0$ are solutions of the ordinary differential equation

(6) $$\dot{V} = V - V_0, \qquad \dot{p} = p - p_0 - 2\phi'(V).$$

Using $\phi'(V_0)=0$ in (5), observe that the point (V_0, p_0) is a repelling rest point of these equations; we let R denote the set of points in the positive (V, p) quadrant which are in the domain of repulsion.

Note that if $\phi \equiv 0$ then R is the whole positive quadrant. Also with the ϕ of m.h.d., R contains precisely those points of the quadrant which lie on the same side of the line $V = V_1$ as does V_0. Namely, given any such point (V, p), the V component of the solution goes to V_0 as $t \to -\infty$. This, with $\phi'(V_0)=0$, implies the p component goes to p_0. Also, as the V component tends to V_1, the absolute value of the p component must go to infinity.

Now let γ be any integral curve of (6) in R. The relevant facts about the behavior of s on γ are the following:

(7) *The critical points of $H|\gamma$ and $s|\gamma$ coincide.*

This follows from $dH = \omega + T\,ds$.

(8) *Any critical point of $s|\gamma$ is a maximum; in particular, $s|\gamma$ has at most one critical point.*

To see this we observe there are two ways of writing the derivative, $\dot{p}$, of p on γ; namely,

$$p - p_0 - 2\phi'(V) = \dot{p} = p_V\dot{V} + p_s\dot{s}.$$

Now differentiate this equation again along γ and evaluate at a critical point of s ($\dot{s}=0$). Writing 0 for terms involving $\dot{s}$ we have

$$p_V\ddot{V} + 0 - 2\phi''(V)\dot{V} = p_{VV}\dot{V}^2 + 0 + p_V\ddot{V} + 0 + p_s\ddot{s}.$$

From (6) we see that $\ddot{V}=\dot{V}=V-V_0$ and so we conclude that

$$\ddot{s} = -p_s^{-1}[2\phi''(V - V_0) + p_{VV}(V - V_0)^2].$$

Now from (4) and (5), $\ddot{s}<0$ so (8) is proved.

Weyl's statement now takes the form:

(9) THEOREM. *If* $(V,p)\neq(V_0,p_0)$ *is a point in* R *at which* $H=0$ *and* $dH\neq 0$, *then* $s|_{H=0}$ *is not critical at* (V,p).

PROOF. Since $dH\neq 0$ at (V,p), the set $\{H=0\}$ meets a neighborhood of (V,p) in a curve. Suppose the restriction of s to this curve is critical at (V,p); then, since $dH=\omega+T\,ds$, the restriction of s to the integral curve of (6) through (V,p) is also critical at (V,p). Consider the negative half orbit, say γ, from (V_0,p_0) to (V,p) (definition of R): H is zero at both ends of γ so $H|\gamma$ is critical at some point of γ strictly between (V_0,p_0) and (V,p). Since H and s are critical together on γ in (7) we find that s has two critical points on an orbit of (6). This contradicts (8).

In order not to spoil the proof, we have not investigated the question of existence of points in $\{H=0\}$ at which $dH=0$. In the case of a perfect gas, this does not happen.

REFERENCES

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