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HOLOMORPHIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS NEAR SINGULAR POINTS

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ABSTRACT. Functional analysis techniques are used to prove a theorem, analogous to the Harris-Sibuya-Weinberg theorem for ordinary differential equations, which yields as corollaries a number of existence theorems for holomorphic solutions of linear functional differential systems of the form $z^D y'(z)=A(z)y(z)+B(z)y(\alpha z)+C(z)y'(\alpha z)$ in the neighborhood of the singularity at $z=0$.

The existence of holomorphic solutions of ordinary differential systems near a singular point has been extensively studied. F. Lettenmeyer [6] showed that a linear system with an irregular singular point at $z=z_0$ may have several linearly independent solutions holomorphic at z_0; his theorem gives an estimate on the number of such solutions. Lettenmeyer's original proof was quite involved; the proof has been greatly simplified by W. A. Harris, Jr., Y. Sibuya, and L. Weinberg [5], who used functional analysis techniques to establish a theorem which includes Lettenmeyer's theorem and several results on systems of Briot-Bouquet type as simple corollaries.

Several authors ([1], [2], [3], [7]) have studied existence of solutions of functional differential equations with contracting arguments in the neighborhood of a singularity at the origin. All the equations considered in their articles are of Briot-Bouquet type, and only Grudo in [3] deals with systems of neutral-differential equations. In this paper we extend the results of Harris, Sibuya and Weinberg to a class of neutral-differential systems, obtaining as corollaries an analogue of Lettenmeyer's theorem and a generalization of the results of Grudo. Our principal result is the following theorem.

THEOREM. *Let $A(z)$, $B(z)$, and $C(z)$ be $n\times n$ matrices holomorphic at $z=0$, let $D=\operatorname{diag}(d_1,\cdots,d_n)$ with nonnegative integers d_i, and let α, $|\alpha|<1$, be a complex constant. Then for every positive integer N sufficiently large, and every polynomial $\phi(z)$ with $z^D\phi(z)$ of degree N, there exists a polynomial $f(z;\phi)$ (depending on A, B, C, α, and N) of degree*

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$N-1$ *such that the linear neutral-differential system*

$$z^D y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z) + f(z;\phi) \tag{1}$$

has a solution $y(z)$ *holomorphic at* $z=0$ *Further,* f *and* y *are linear and homogeneous in* ϕ, *and* $z^D(y-\phi)=O(z^{N+1})$ *as* $z\to 0$.

PROOF. The proof is an application of the Banach fixed point theorem. Let $\delta>0$ and let X be the set of all n-vector valued functions $f=f(z)$ whose components have absolutely convergent power series expansions in $|z|\leqq\delta$. For $f\in X$, $f(z)=\sum_{k=0}^{\infty} f_k z^k$, $f_k=(f_k^1,\cdots,f_k^n)^T$, define $\|f\|=\sum_{k=0}^{\infty}|f_k|\delta^k$, where $|f_k|=\sum_{j=1}^{n}|f_k^j|$. With this norm, X is a Banach space.

For a sufficiently large positive integer N, define the mapping $L_N: X\to X$ as follows: $L_N y=g$, where

$$y(z) = (y^1(z),\cdots,y^n(z))^T, \qquad g(z) = (g^1(z),\cdots,g^n(z))^T,$$

with $y^j(z)=\sum_{k=0}^{\infty} y_k^j z^k$, $g^j(z)=\sum_{k=N}^{\infty}(y_k^j z^{k+1-d_j})/(k+1-d_j)$. Hence

$$\|L_N y\| \leqq \sum_{j=1}^{n} \frac{\delta^{1-d_j}}{N+1-d_j}\|y\|. \tag{2}$$

Define $\hat{y}(z)=(y^1(\alpha z),\cdots,y^n(\alpha z))^T\equiv(\hat{y}^1(z),\cdots,\hat{y}^n(z))^T$ with

$$\hat{y}^j(z) = \sum_{k=0}^{\infty}\hat{y}_k z^k \equiv \sum_{k=0}^{\infty} y_k^j \alpha^k z^k.$$

Also define $y^*(z)=(y^{*1}(z),\cdots,y^{*n}(z))^T$, with

$$y^{*j}(z) = \sum_{k=0}^{\infty}(k+1)\alpha^k y_{k+1}^j z^k.$$

Note that $\hat{y}$ and y^* have absolutely convergent power series expansions for $|z|\leqq\delta$, and also that

$$\|\hat{y}\| \leqq \|y\|. \tag{3}$$

Furthermore, setting $\chi(z)=\sum_{k=0}^{\infty}(\sum_{j=1}^{n}|y_k^j|)z^k$, $|z|\leqq\delta$, we have

$$\chi'(|\alpha|\, z) = \sum_{k=1}^{\infty} k\left(\sum_{j=1}^{n}|y_k^j|\right)|\alpha|^{k-1} z^{k-1}, \qquad |z|\leqq\delta.$$

By the Cauchy integral formula,

$$|\chi'(|\alpha|\, z)| \leqq \frac{\max_{|\zeta|=\delta}|\chi(\zeta)|}{\delta^2(1-|\alpha|)^2} = \frac{\|y\|}{\delta^2(1-|\alpha|)^2}, \qquad |z|\leqq\delta.$$

Hence

$$\|y^*\| = |\chi'(|\alpha|\,\delta)| \leqq \|y\|/\delta^2(1-|\alpha|)^2. \tag{4}$$

If M is an $n\times n$ matrix, $M=(m^{ij})$, with elements having absolutely convergent power series expansions for $|z|\leqq\delta$, $m^{ij}=\sum_{k=0}^{\infty} m_k^{ij}z^k$, then for $f\in X$ we have $Mf\in X$ and $\|Mf\|\leqq\|M\|\ \|f\|$, where

$$\|M\| = \sum_{i,j=1}^{n} \left(\sum_{k=0}^{\infty} |m_k^{ij}|\,\delta^k\right).$$

Let $\phi=(\phi^1,\cdots,\phi^n)^T$ be a vector polynomial with $\phi^j(z)=\sum_{k=0}^{N-d_j}\phi_k^j z^k$, and consider the functional equation in X

$$y = \phi + T_N[y], \tag{5}$$

where $T_N[y]=L_N(Ay+B\hat{y}+Cy^*)$. The estimates (2)–(4) imply that for N sufficiently large, $\|T_N\|<1$, and thus there exists a unique solution $y\in X$, $y(\cdot\,;\phi)=(I-T_N)^{-1}\phi$.

From the definition of the mapping T_N it follows that the holomorphic solution of the functional equation (5) satisfies the linear differential system of the form (1), where

$$\begin{aligned} f(z;\phi) &\equiv \sum_{k=0}^{N-1} f_k z^k \\ &\equiv z^D\frac{d\phi}{dz} - \sum_{k=0}^{N-1} Ay(\cdot\,;\phi)_k z^k - \sum_{k=0}^{N-1} B\hat{y}(\cdot\,;\phi)_k z^k - \sum_{k=0}^{N-1} Cy^*(\cdot\,;\phi)_k z_k. \end{aligned} \tag{6}$$

Since the coefficients of $y(\cdot\,;\phi)$ (and thus also $\hat{y}$ and y^*) are linear in the coefficients of ϕ, this is also true for the f_k. The proof is complete.

The corollaries below follow from the theorem similarly to the proofs of corresponding results in [5].

COROLLARY 1. *Let $D=$ trace D and $n-d\geqq 0$. Then the system*

$$z^D y'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z) \tag{7}$$

has at least $n-d$ linearly independent solutions holomorphic at $z=0$.

COROLLARY 2. *Let*

$$A(z) = \sum_{k=0}^{\infty} A_k z^k, \quad B(z) = \sum_{k=0}^{\infty} B_k z^k, \quad \textit{and} \quad C(z)=\sum_{k=1}^{\infty} C_k z^k$$

be convergent for $|z|<a$ $(a>0)$, and let $y(z)=\sum_{k=0}^{\infty} y_k z^k$ be a formal solution of

$$zy'(z) = A(z)y(z) + B(z)y(\alpha z) + C(z)y'(\alpha z). \tag{8}$$

Then $y(z)$ is convergent for $|z|<a$.

COROLLARY 3. *Let A, B, and C be as in Corollary 2, let m be a fixed integer, let $\alpha \neq 0$, and let n_{m+k} be the number of linearly independent eigenvectors corresponding to the eigenvalue $m+k$ of the matrix*

$$\mathfrak{A}_{m+k} = [A_0 + \alpha^{m+k} B_0 + (m + k)\alpha^{m+k-1} C_1].$$

Then the number N_m ($\geqq 0$) of linearly independent solutions of the differential system (8) of the form $y = \sum_{k=0}^{\infty} y_k z^{m+k}$ satisfies $N_m \leqq n_m + n_{m+1} + \cdots$. If, in addition, $B_0 = C_1 = 0$, then $N_m \geqq \max(n_m, n_{m+1}, \cdots)$.

REMARKS. 1. The results extend without change to systems with several deviating arguments of the same form.

2. If $A_0 = B_0 = C_0 = 0$, then $z = 0$ is an ordinary point for the system (8). Hence, by Corollary 1, there exist at least n linearly independent solutions for this system. If, in addition, $C_1 = 0$, then the coefficients of each formal solution are determined recursively and there exist exactly n linearly independent solutions of the system holomorphic at $z = 0$.

3. Analogous results hold for nonlinear systems of the form

$$z^D y'(z) = h(z, y(z), y(\alpha z), y'(\alpha z)) + f(z; \phi)$$

and can be obtained by considerations similar to those in the paper of Harris [4].

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