PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

TWO LIFTING THEOREMS

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ABSTRACT. It is assumed that the measure algebra involved has cardinality $2^{\aleph_0}$, and it is assumed further that $2^{\aleph_0}=\aleph_1$. Then liftings exist when the σ-field is not necessarily complete, and strong Borel liftings exist in the locally compact σ-compact metric case.

1. **Introduction.** Let $(X_0, \mathscr{F}_0, \mu_0)$ be a probability space, let $B(X_0, \mathscr{F}_0)$ be the Banach algebra of bounded real $\mathscr{F}_0$ measurable functions on X_0, the norm being $\|f\|=\sup_x\{|f(x)|:x \in X_0\}$, and let $\mathscr{I}_0=\{f \in B(X_0, \mathscr{F}_0): \int |f|\, d\mu_0=0\}$ be the closed ideal of μ_0-null functions. The quotient Banach algebra $B(X_0, \mathscr{F}_0)/\mathscr{I}_0$ may be identified as the familiar $L_\infty(X_0, \mathscr{F}_0, \mu_0)$; let $q_0:B(X_0, \mathscr{F}_0)\to L_\infty(X_0, \mathscr{F}_0, \mu_0)$ be the quotient mapping. A *lifting* $\Lambda_0: L_\infty(X_0, \mathscr{F}_0, \mu_0)\to B(X_0, \mathscr{F}_0)$ is a selection of representative $\Lambda_0(f+\mathscr{I}_0) \in f+\mathscr{I}_0$ from each equivalence class $f+\mathscr{I}_0, f \in B(X_0, \mathscr{F}_0)$, in such a way that the representatives constitute a subalgebra of $B(X_0, \mathscr{F}_0)$; it is required also that $\Lambda_0(1+\mathscr{I}_0)=1$. That is, Λ_0 is an algebraic cross section of q_0 which preserves the unit: Λ_0 is an algebraic homomorphism, $q_0\Lambda_0=$ (identity), $\Lambda_0(1+\mathscr{I}_0)=1$. The proof in **[1]** that liftings exist requires that $\mathscr{F}_0$ be complete with respect to μ_0. We show that $\mathscr{F}_0$ need not be complete provided: (i) the measure algebra $(\mathscr{F}_0, \mu_0)$ has cardinal $2^{\aleph_0}$; (ii) $2^{\aleph_0}=\aleph_1$ (the continuum hypothesis).

Suppose further that X_0 is a topological space, the bounded continuous real functions $C_b(X_0)$ are $\mathscr{F}_0$ measurable, and $\mu_0(U)>0$ for open $U\neq\varnothing$. A *strong lifting* is a lifting Λ_0 such that $\Lambda_0(f+\mathscr{I}_0)=f, f \in C_b(X_0)$. Various sufficient conditions are known [1] for the existence of strong liftings, e.g., X_0 locally compact σ-compact metric. We prove here (with $2^{\aleph_0}=\aleph_1$) that strong Borel liftings exist in this last case; that is, strong liftings such that each representative is measurable with respect to the uncompleted σ-field of Borel subsets of locally compact σ-compact metric X_0.

2. **Representation spaces.** Under the Gel'fand representation; $B(X_0, \mathscr{F}_0)$ is isometrically algebraically isomorphic to the continuous real functions

Presented to the Society, August 23, 1973; received by the editors January 1, 1973 and, in revised form, April 26, 1973.

AMS (MOS) subject classifications (1970). Primary 46G15.

Key words and phrases. Lifting theory.

$C(W)$ on a certain compact Hausdorff space W; let $\iota: C(W) \to B(X_0, \mathscr{F}_0)$ be the inverse isomorphism. This has adjoint $\iota^*: ba(X_0, \mathscr{F}_0) \to \mathrm{rca}(W)$ when we identify the conjugate Banach space $[B(X_0, \mathscr{F}_0)]^*$ with the space $ba(X_0, \mathscr{F}_0)$ of bounded finitely additive set functions on σ-field $\mathscr{F}_0$, and the conjugate Banach space $[C(W)]^*$ with the space $\mathrm{rca}(W)$ of regular Borel measures on W. We will assume without essential loss of generality that $B(X_0, \mathscr{F}_0)$ separates X_0, i.e., if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ for some $f \in B(X_0, \mathscr{F}_0)$. Then with δ_x the point measure at x, the set $\{\iota^*\delta_x : x \in X_0\}$ constitutes a copy of X_0 contained as a dense subset of $W \subset w^*\text{-rca}(W)$; we identify W with the set $\{\delta_w : w \in W\} \subset w^*\text{-rca}(W)$ whenever convenient. A bounded real function f on $X_0 \subset W$ extends to a member of $C(W)$ iff $f \in B(X_0, \mathscr{F}_0)$, and ι is represented as the restriction mapping $\iota g = g|X_0$, $g \in C(W)$. Each $E \in \mathscr{F}_0$ has closure $\mathrm{cl}_W E$ which is open, and the correspondence $E \leftrightarrow \mathrm{cl}_W E$ is 1-1 between $\mathscr{F}_0$ and the open closed subsets of W. The σ-field of Baire subsets of W is generated by $\{\mathrm{cl}_W E : E \in \mathscr{F}_0\}$, and if $\theta \in ba(X_0, \mathscr{F}_0)$ is given then $\iota^*\theta \in \mathrm{rca}(W)$ is determined on the Baire subsets of W by $(\iota^*\theta)(\mathrm{cl}_W E) = \theta(E)$, $E \in \mathscr{F}_0$, and then on the Borel sets by regularity.

As a Banach lattice, $B(X_0, \mathscr{F}_0)$ is boundedly σ-complete ($=\aleph_0$-reticulated): any countable subset of $B(X_0, \mathscr{F}_0)$ bounded above has a supremum in $B(X_0, \mathscr{F}_0)$; the isomorphic $C(W)$ enjoys the same property. Dually, W is basically disconnected: disjoint open Baire subsets of W have disjoint open closures [3, Chapter VII]. Equivalently, the interior F° of any closed Baire set F is closed, so that a closed Baire set is of the form $(\mathrm{cl}_W E) \cup N$ with $E \in \mathscr{F}_0$ and N a closed nowhere dense Baire set. The following results from [2] will be used. If $N \subset W$ is a closed nowhere dense Baire set then $N = \lim_n \mathrm{cl}_W E_n$ for some sequence $E_1 \supset E_2 \supset \cdots$ in $\mathscr{F}_0$ such that $\lim_n E_n = \varnothing$. If $\theta \in ba(X_0, \mathscr{F}_0)$ is countably additive on $\mathscr{F}_0$ then $(\iota^*\theta)(N) = 0$ for every closed nowhere dense Baire set N.

Let $\mu = \iota^*\mu_0 \in \mathrm{rca}(W)$ correspond to μ_0, and let X be the closed support of μ in W. The ideal $\mathscr{I} = \iota^{-1}\mathscr{I}_0 \subset C(W)$ which corresponds to $\mathscr{I}_0$ is clearly $\{f \in C(W) : f(X) = 0\}$, and the quotient mapping q_0, isomorphic to the quotient mapping $C(W) \to C(W)/\mathscr{I}$, is isomorphic to the restriction mapping $q: C(W) \to C(X)$ given by $qf = f|X$, $f \in C(W)$. Space X is the Gel'fand space of $L_\infty(X_0, \mathscr{F}_0, \mu_0)$, and is hyperstonian with μ as category measure. That is, $L_\infty(X_0, \mathscr{F}_0, \mu_0)$ is isometrically algebraically isomorphic to $C(X)$, X is extremally disconnected, and $\mu(A) = \mu(A^\circ) > 0$ if $A^\circ \neq \varnothing$, Borel $A \subset X$. We denote by $\mathscr{F}$ the class of open closed subsets of X; the sets $\{X \cap \mathrm{cl}_W E : E \in \mathscr{F}_0\}$ (not necessarily distinct) comprise $\mathscr{F}$. The measure algebra $(\mathscr{F}_0, \mu_0)$ is isomorphic to the quotient Boolean algebra $\mathscr{F}/$(nowhere dense sets).

The following relation between the closure operators in W and X is needed in the proof of Theorem 1.

LEMMA 1. *If $U \subset W$ is an open Baire set then $X \cap \mathrm{cl}_W U = \mathrm{cl}_X(X \cap U)$.*

PROOF. We have $\Phi = \mathrm{cl}_W U = U \cup N$ with Φ open closed and N a closed nowhere dense Baire set in W. Since $\mu(X \cap N) = 0$ and μ is category relative to X, $X \cap N$ is nowhere dense in X. The closure $\Theta = \mathrm{cl}_X(X \cap U) = (X \cap U) \cup N_1$ of open $X \cap U$ is open closed in extremally disconnected X. Thus

$$\Theta = [X \cap (\Phi - N)] \cup N_1 = (X \cap \Phi) \Delta [(X \cap N) - N_1],$$

and since Θ and $X \cap \Phi$ are each open closed and $X \cap N$ is nowhere dense, $\Theta = X \cap \Phi$. $\square$

3. **Partial liftings.** To a lifting $\Lambda_0 : L_\infty(X_0, \mathscr{F}_0, \mu_0) \to B(X_0, \mathscr{F}_0)$ there corresponds an algebraic homomorphism $\Lambda : C(X) \to C(W)$ with the properties $q\Lambda = (\text{identity})$, $\Lambda 1 = 1$; we call Λ a lifting also. The adjoint $\Lambda^* : \mathrm{rca}(W) \to \mathrm{rca}(X)$ restricts to a mapping $\lambda : W \to X$ which is a retraction of W onto X. Conversely, such a retraction determines a lifting according to: $(\Lambda f)(w) = f(\lambda w)$, $w \in W$, $f \in C(X)$.

We denote by $\mathscr{A}$ the set $\mathscr{A} = \{\alpha \subset C(X) : \alpha$ is a closed subalgebra of $C(X)$ containing the constants$\}$. Each $\alpha \in \mathscr{A}$ is isometrically algebraically isomorphic to $C(Z_\alpha)$ for a certain compact Hausdorff space Z_α. If $j_\alpha : C(Z_\alpha) \to C(X)$ is the injection onto α then $j_\alpha^* : \mathrm{rca}(X) \to \mathrm{rca}(Z_\alpha)$ restricts to the quotient mapping $v_\alpha : X \to Z_\alpha$ associated with α, i.e., $(j_\alpha f)(x) = f(v_\alpha x)$, $x \in X$, $f \in C(Z_\alpha)$.

By a partial lifting $\Lambda_\alpha : \alpha \to C(W)$ we will mean an algebraic homomorphism defined only on the subalgebra α of $C(X)$ with the properties $q\Lambda_\alpha = (\text{identity})$, $\Lambda_\alpha 1 = 1$. Equivalent to Λ_α is the algebraic homomorphism $\tilde{\Lambda}_\alpha : C(Z_\alpha) \to C(W)$ given by $\tilde{\Lambda}_\alpha = \Lambda_\alpha j_\alpha$ and such that $q\tilde{\Lambda}_\alpha = j_\alpha$, $\tilde{\Lambda}_\alpha 1 = 1$. The adjoint $\tilde{\Lambda}_\alpha^* : \mathrm{rca}(W) \to \mathrm{rca}(Z_\alpha)$ restricts to the partial retraction $\lambda_\alpha : W \to Z_\alpha$ dual to $\tilde{\Lambda}_\alpha$ and Λ_α, i.e., $\lambda_\alpha | X = v_\alpha$. Since $\tilde{\Lambda}_\alpha$ and j_α^{-1} are isometries, $\Lambda_\alpha = \tilde{\Lambda}_\alpha j_\alpha^{-1}$ is an isometry.

Let $\mathscr{L}$ denote the family $\mathscr{L} = \{\Lambda_\alpha : \alpha \in \mathscr{A}$ and Λ_α is a partial lifting with domain $\alpha\}$. An order $<$ in $\mathscr{L}$ is defined by: $\Lambda_\alpha < \Lambda_\beta$ iff Λ_β extends Λ_α; that is, $\alpha \subset \beta$ and $\Lambda_\beta | \alpha = \Lambda_\alpha$.

LEMMA 2. *Any ascending chain in $\mathscr{L}$ has an upper bound in $\mathscr{L}$.*

PROOF. Suppose $\{\Lambda_{\alpha_\nu} : \nu \in M\}$ is an ascending chain in $\mathscr{L}$: M is a totally ordered indexing set and $\Lambda_{\alpha_\mu} < \Lambda_{\alpha_\nu}$ for $\mu \leqq \nu \in M$. Define subalgebra γ of $C(X)$ as $\gamma = \bigcup_\nu \{\alpha_\nu : \nu \in M\}$, and let $\alpha \in \mathscr{A}$ be the closure in $C(X)$ of γ. An operator $\Lambda_\gamma : \gamma \to C(W)$ is defined consistently on γ by the family of its restrictions $\Lambda_\gamma|\alpha_\mu=\Lambda_{\alpha\mu}$, $\mu\in M$, and one verifies easily that Λ_γ is an algebraic homomorphism with the properties $q\Lambda_\gamma=$(identity), $\Lambda_\gamma 1=1$. Since each $\Lambda_{\alpha\mu}$, $\mu\in M$, is an isometry, Λ_γ is an isometry, and so extends uniquely by continuity to an operator $\Lambda_\alpha:\alpha\to C(W)$. By continuity, Λ_α is an algebraic homomorphism with the properties $q\Lambda_\alpha=$(identity), $\Lambda_\alpha 1=1$; that is, $\Lambda_\alpha\in\mathscr{L}$. Since $\Lambda_{\alpha\mu}<\Lambda_\alpha$, $\mu\in M$, by construction, Λ_α is the upper bound sought. □

Suppose $\alpha\in\mathscr{A}$ and $X_1\in\mathscr{F}$ are given, and let $\beta\in\mathscr{A}$ be the algebra generated by $\{\alpha,\chi_{X_1}\}$. With $X_2=X-X_1$, β consists of all $f\in C(X)$ of the form $f=f_1\chi_{X_1}+f_2\chi_{X_2}$ for some $f_1,f_2\in\alpha$. We may rewrite this as $f=(f_1+\mathscr{I}_1)\chi_{X_1}+(f_2+\mathscr{I}_2)\chi_{X_2}$ where $\mathscr{I}_i=\{g\in\alpha:g\chi_{X_i}=0\}$, $i=1,2$, are closed ideals in α; the elements $(f_i+\mathscr{I}_i)\in\alpha/\mathscr{I}_i$, $i=1,2$, are then uniquely determined by $f\in\beta$. The space Z_β associated with β is the free union $Z_\beta=Z_{\alpha 1}\cup Z_{\alpha 2}$ of copies of the subsets $Z_{\alpha i}=v_\alpha X_i$, with $\alpha/\mathscr{I}_i$ isomorphic to $C(Z_{\alpha i})$, $i=1,2$.

Lemma 3. *Suppose $\Lambda_\alpha\in\mathscr{L}$ and X_1, $X_2=X-X_1\in\mathscr{F}$ are given, and let $\beta\in\mathscr{A}$ be the algebra generated by $\{\alpha,\chi_{X_1}\}$. If $\Lambda_\beta\in\mathscr{L}$ exists such that $\Lambda_\alpha<\Lambda_\beta$ then open closed subsets W_1 and $W_2=W-W_1$ of W are determined such that*

(i) $\Lambda_\beta f=(\Lambda_\alpha f_1)\chi_{W_1}+(\Lambda_\alpha f_2)\chi_{W_2}$ *for* $f=(f_1+\mathscr{I}_1)\chi_{X_1}+(f_2+\mathscr{I}_2)\chi_{X_2}\in\beta$,

(ii) $W_i\cap X=X_i$, $i=1,2$,

(iii) $W_i\subset\lambda_\alpha^{-1}v_\alpha X_i$, $i=1,2$.

Conversely, if open closed W_1 and $W_2=W-W_1$ in W are given satisfying (ii) *and* (iii) *then* (i) *serves to define $\Lambda_\beta\in\mathscr{L}$ such that $\Lambda_\alpha<\Lambda_\beta$.*

Proof. Suppose $\Lambda_\beta\in\mathscr{L}$ is given such that $\Lambda_\alpha<\Lambda_\beta$. From $\chi^2_{X_1}=\chi_X$ and $\chi_{X_2}=1-\chi_{X_1}$, and the fact that Λ_β is an algebraic homomorphism such that $\Lambda_\beta 1=1$, we find that $\Lambda_\beta\chi_{X_i}=\chi_{W_i}$, $i=1,2$, for open closed subsets W_1 and $W_2=W-W_1$ of W. Since $\mathscr{I}_i\subset\alpha$, $\chi_{X_i}\in\beta$, and $\mathscr{I}_i\chi_{X_i}=0$, $i=1,2$, we must have

$$\Lambda_\beta(\mathscr{I}_i\chi_{X_i}) = (\Lambda_\alpha\mathscr{I}_i)(\Lambda_\beta\chi_{X_i}) = (\Lambda_\alpha\mathscr{I}_i)\chi_{W_i} = 0, \qquad i=1,2;$$

this is condition (iii). The property $q\Lambda_\beta=$(identity) gives condition (ii). The converse arguments are similar. □

Theorem 1. *Suppose $\Lambda_\alpha\in\mathscr{L}$ and $X_1\in\mathscr{F}$ are given, and let $\beta\in\mathscr{A}$ be the subalgebra generated by $\{\alpha,\chi_{X_1}\}$. If Z_α is metrizable then partial liftings Λ_β exist which extend Λ_α.*

Proof. The conditions $W_i\subset\lambda_\alpha^{-1}v_\alpha X_i$, $i=1,2$, of Lemma 3 are equivalent to $W_i\supset\lambda_\alpha^{-1}U_i$, $i=1,2$, where $U_i=Z_\alpha-v_\alpha X_{3-i}$, $i=1,2$, are disjoint open subsets of Z_α. If Z_α is metrizable the Borel sets are Baire sets, U_1 and U_2 are disjoint open Baire sets in Z_α, whence $\lambda_\alpha^{-1}U_1$ and $\lambda_\alpha^{-1}U_2$ are disjoint open Baire sets in W. Using the fact that W is basically disconnected, we have that $\Phi_1 = \mathrm{cl}_W(\lambda^{-1}U_1)$ and $\Phi_2 = \mathrm{cl}_W(\lambda^{-1}U_2)$ are disjoint open closed subsets of W.

By Lemma 1,

$$X \cap \Phi_1 = \mathrm{cl}_X(X \cap \lambda_\alpha^{-1}U_1) = \mathrm{cl}_X(v_\alpha^{-1}U_1) = \mathrm{cl}_X(X - v_\alpha^{-1}v_\alpha X_2) \subset X_1,$$

since $X - v_\alpha^{-1}v_\alpha X_2 \subset X_1$ and X_1 is closed; similarly, $\Phi_2 \cap X \subset X_2$.

Let Γ_1 and $\Gamma_2 = W - \Gamma_1$ be any open closed subsets of W such that $X \cap \Gamma_i = X_i$, $i=1,2$. With $\Theta = W - (\Phi_1 \cup \Phi_2)$ open closed, define open closed W_1 and W_2 by $W_i = \Phi_i \cup (\Theta \cap \Gamma_i)$, $i=1,2$. It is clear that $W_2 = W - W_1$. Since $W_i \cap X = (\Phi_i \cap X) \cup (\Theta \cap X_i) \subset X_i$ and $\{W_1, W_2\}$, $\{X_1, X_2\}$ are partitions, we have necessarily $W_i \cap X = X_i$, $i=1,2$. Conditions (ii) and (iii) of Lemma 3 being satisfied, (i) gives the extension sought. □

4. **The Lifting theorems.** The cardinal of the measure algebra $(\mathscr{F}_0, \mu_0)$ is either finite or at least $2^{\aleph_0}$; we assume from now on that the cardinal is $2^{\aleph_0}$. We assume further that $2^{\aleph_0} = \aleph_1$, and we let $\{F_\nu : \nu < \aleph_1\}$ be a well ordering of the elements of $\mathscr{F}$.

THEOREM 2 (INCOMPLETE LIFTING THEOREM). *If the measure algebra $(\mathscr{F}_0, \mu_0)$ has cardinal $2^{\aleph_0} = \aleph_1$ then liftings $\Lambda_0 : L_\infty(X_0, \mathscr{F}_0, \mu_0) \to B(X_0, \mathscr{F}_0)$ exist.*

PROOF. The parts of the transfinite induction are:

(i) α_0 is the constants, Z_{α_0} is a singleton, $\Lambda_{\alpha_0} 1 = 1$.

(ii) For $\nu < \aleph_1$ a successor ordinal, suppose $\{\Lambda_{\alpha_\gamma} : \gamma < \nu\}$ is an ascending chain in $\mathscr{L}$ such that each Z_{α_γ}, $\gamma < \nu$, is metrizable; in particular, $Z_{\alpha_{\nu-1}}$ is metrizable. Let α_ν be the algebra generated by $\{\alpha_{\nu-1}, \chi_{F_{\nu-1}}\}$, and let Λ_{α_ν} be the partial lifting provided by Theorem 1. It is clear that Z_{α_ν} is metrizable, so that $\{\Lambda_{\alpha_\gamma} : \gamma < \nu+1\}$ is an ascending chain in $\mathscr{L}$ such that each Z_{α_γ} is metrizable, $\gamma < \nu+1$.

(iii) For $\nu < \aleph_1$ a limit ordinal, suppose $\{\Lambda_{\alpha_\gamma} : \gamma < \nu\}$ is an ascending chain in $\mathscr{L}$ such that each Z_{α_γ}, $\gamma < \nu$, is metrizable. Lemma 2 provides Λ_{α_ν} on $\alpha_\nu = \mathrm{cl}_{C(X)} \bigcup_{\gamma<\nu} \alpha_\gamma$ such that $\{\Lambda_{\alpha_\gamma} : \gamma < \nu+1\}$ is an ascending chain in $\mathscr{L}$. If $\sigma_\gamma \subset \alpha_\gamma$ is a countable set dense in α_γ, $\gamma < \nu$, then $\bigcup_{\gamma<\nu} \sigma_\gamma \subset \alpha_\nu$ is a countable set dense in α_ν, so that Z_{α_ν} is metrizable.

By transfinite induction, there exists an ascending chain $\{\Lambda_{\alpha_\gamma} : \gamma < \aleph_1\}$, and Lemma 2 provides an ascending chain $\{\Lambda_{\alpha_\gamma} : \gamma \leqq \aleph_1\}$. The algebra $\alpha_{\aleph_1} = \bigcup_{\nu<\aleph_1} \alpha_\nu \in \mathscr{A}$ contains every χ_{F_ν}, $\nu < \aleph_1$, and so is all of $C(X)$. Thus the partial lifting $\Lambda_{\alpha_{\aleph_1}}$ is a lifting. □

THEOREM 3 (STRONG BOREL LIFTING THEOREM). *Let X_0 be a locally compact σ-compact metric space, let $\mathscr{F}_0$ be the Borel subsets of X_0, let μ_0 be strictly positive on nonempty open sets, and assume $2^{\aleph_0} = \aleph_1$. Then* *liftings* $\Lambda_0: L_\infty(X_0, \mathscr{F}_0, \mu_0) \to B(X_0, \mathscr{F}_0)$ *exist such that* $\Lambda_0(f+\mathscr{I}_0)=f$, $f \in C_b(X_0)$.

PROOF. With $C_0(X_0)$ the continuous real functions vanishing at infinity, let $A_0 \subset C_b(X_0)$ be the algebra generated by $\{C_0(X_0), 1\}$; A_0 is isometrically algebraically isomorphic to $C(Z_{\alpha_0})$ where $Z_{\alpha_0}=X_0 \cup \{\infty\}$ is the one point compactification of X_0 if X_0 is noncompact, or $Z_{\alpha_0}=X_0$ if X_0 is compact. The assumption that μ_0 is strictly positive on nonempty open sets implies that for each $f \in A_0$, f is the unique continuous function in the class $f+\mathscr{I}_0 \in L_\infty(X_0, \mathscr{F}_0, \mu_0)$. Equivalently, a partial lifting $\Lambda_{\alpha_0}: \alpha_0 \to C(W)$ of the subalgebra $\alpha_0 = q\iota^{-1}A_0 \subset C(X)$ is determined such that $\Lambda_{\alpha_0} q\iota^{-1} f = \iota^{-1} f$, $f \in A_0$. The space Z_{α_0} associated with α_0 is the one defined above, and the assumption that X_0 is σ-compact implies that Z_{α_0} is metrizable.

We now apply transfinite induction; parts (ii) and (iii) are as in the proof of Theorem 2, but part (i) is: $\alpha_0 = q\iota^{-1}A_0$, Z_{α_0} and Λ_{α_0} as just described. We obtain a lifting $\Lambda: C(X) \to C(W)$ such that $\Lambda_{\alpha_0} < \Lambda$; the isomorphic $\Lambda_0: L_\infty(X_0, \mathscr{F}_0, \mu_0) \to B(X_0, \mathscr{F}_0)$ is such that $\Lambda_0(f+\mathscr{I}_0)=f$, $f \in A_0$.

If X_0 is compact we are done; suppose X_0 is noncompact. Since X_0 is assumed to be σ-compact, there exists $h \in C_0(X_0)$ such that $h(x)>0$, $x \in X_0$. From

$$\Lambda_0(hf + \mathscr{I}_0) = [\Lambda_0(h + \mathscr{I}_0)][\Lambda_0(f+\mathscr{I}_0)], \qquad f \in B(X_0, \mathscr{F}_0),$$

and $\Lambda_0(h+\mathscr{I}_0)=h>0$ we have

$$\Lambda_0(f + \mathscr{I}_0) = h^{-1}\Lambda_0(hf + \mathscr{I}_0), \qquad f \in B(X_0, \mathscr{F}_0).$$

If $f \in C_b(X_0)$ then $hf \in C_0(X_0)$ and $\Lambda_0(hf+\mathscr{I}_0)=hf$, giving $\Lambda_0(f+\mathscr{I}_0)=f$, $f \in C_b(X_0)$. That is, Λ_0 is a strong Borel lifting. □

We conclude with the following remarks. In the proof of the lifting theorem given in [**1**] it is required that the subalgebras α involved in the partial liftings be boundedly complete; that is, the Z_α are extremally disconnected. In the induction step corresponding to Theorem 1 of the present paper the sets $U_1, U_2 \subset Z_\alpha$ have closures in Z_α which are disjoint and open closed, hence Baire, and these closures can replace U_1, U_2 in the argument. The induction step corresponding to Lemma 2 becomes much more difficult, however. The partial liftings Λ_{α_ν}, $\nu \in M$, must be extended not only to our $\alpha = \mathrm{cl}_{C(X)}[\bigcup \{\alpha_\nu : \nu \in M\}]$ (this is the elementary L_∞ martingale theorem given above) but to the boundedly complete algebra generated by α; this requires the completeness of $\mathscr{F}_0$ with respect to μ_0 [**1**, Theorem IV. 2].

ADDED IN PROOF. Theorem 2 of the present paper, but not Theorem 3, can be derived from the results of [**4**].

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