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TOPOLOGICAL ALGEBRAS WITH A GIVEN DUAL

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ABSTRACT. Given an algebra E and a total subspace E' of its algebraic dual, we obtain necessary and sufficient conditions in terms of E' for the existence of an A-convex or a locally m-convex topology on E compatible with duality (E, E'). It has also been proved that if E with the weak topology $w(E, E')$ is the closed linear hull of a bounded set and has hypocontinuous multiplication then it is locally m-convex.

1. **Introduction.** Let E be a complex (or real) algebra and E' be a total subspace of the algebraic dual E^*. To avoid repetitions we use the notation, terminology and results in [3] and [4] without specifications. An algebra with a locally convex linear topology for which multiplication is separately continuous will be called a *locally convex algebra*. An absolutely convex set B in E is called *right* (*left*) *A-convex* if it absorbs Bx (xB) for each $x \in E$, it will be called *A-convex* if it is both right and left A-convex. A locally convex algebra is called (*right*, *left*) *A-convex* if there exists a basis of (right, left) A-convex neighbourhoods of zero. Multiplication in a locally convex algebra will be said to be *right* (*left*) *hypocontinuous* if given a neighbourhood U of o and a bounded set B there exists a neighbourhood V of o satisfying $VB \subset U$ ($BV \subset U$). We say that multiplication is *hypocontinuous* if it is both right and left hypocontinuous. Gulick [5] has, however, called right hypocontinuity by hypocontinuity.

In §2 we answer the following question asked by Cochran [4].

(3.7) Under what conditions, in terms of E', does $\Sigma(E, E')$ or $\chi(E, E')$—the finest A-convex or locally m-convex topology on E compatible with duality (E, E')—exist?

It is known ([3] and [9], MR **41** #7435) that for E with the weak topology $w(E, E')$ the conditions of joint continuity of multiplication, of A-convexity and of local m-convexity are mutually equivalent. We prove

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in §3 that if $(E, w(E, E'))$ is the closed linear hull of a bounded subset of itself then the condition of hypocontinuity of multiplication is also equivalent to all these conditions.

For $y \in E$ and $f \in E^*$, the *right y-multiplicative translate* f_y and the *left y-multiplicative translate* ${}_yf$ of f are given by $f_y(x)=f(xy)$ and ${}_yf(x)=f(yx)$ for $x \in E$ respectively. For $y \in E$ and $S \subset E^*$, let $S(y)=\{f(y): f \in S\}$, $S_y=\{f_y: f \in S\}$ and ${}_yS=\{{}_yf: f \in S\}$.

2. **Topologies on E compatible with duality (E, E').**

(2.1) DEFINITION. A set $S \subset E^*$ is called *collectionwise multiplicative* if $S(xy) \subset S(x)S(y)$ for all $x, y \in E$.

(2.2) DEFINITION. A set $S \subset E^*$ is called *collectionwise right* (*left*) *multiplicative-translation invariant* if for each $y \in E$ there is $\rho_y \geqq 0$ satisfying $S_y(x) \subset \rho_y S(x)$ (${}_yS(x) \subset \rho_y S(x)$) for all $x \in E$. S will be called *collectionwise multiplicative-translation invariant* if it is both collectionwise right and collectionwise left multiplicative-translation invariant.

It is easy to see that every collection of multiplicative linear functionals is collectionwise multiplicative and every balanced, $w(E^*, E)$-bounded, collectionwise multiplicative subset of E^* is collectionwise multiplicative-translation invariant. Also an arbitrary union of collectionwise multiplicative sets is collectionwise multiplicative and a finite union of balanced collectionwise (right, left) multiplicative-translation invariant sets is collectionwise (right, left) multiplicative-translation invariant.

(2.3) LEMMA. *Let $S \subset E'$ be balanced and $w(E', E)$-compact, and let S° be its polar in E.*

(i) *S° is idempotent if and only if S is collectionwise multiplicative.*

(ii) *S° is* (*right, left*) *A-convex if and only if S is collectionwise* (*right, left*) *multiplicative-translation invariant.*

PROOF. (i) Sufficiency is clear.

Necessity. For $x \in E$, let $p(x)=\sup\{|f(x)|: f \in S\}$. Since S is $w(E', E)$-compact, $p(x)<\infty$ and there is an $f \in S$ (depending on x) satisfying $p(x)=|f(x)|$. Because S is balanced, $g=\operatorname{signum} f(x)\cdot f$ is in S. So $p(x)=g(x)$ for some g in S. Also $S^\circ=\{x \in E: p(x) \leqq 1\}$ and p is its Minkowski functional. Now S° is idempotent, so p is submultiplicative i.e. $p(xy) \leqq p(x)p(y)$ for all x, y in E.

Let $x, y \in E$ and $f \in S$. Then $|f(xy)| \leqq p(x)p(y)$. So there is a scalar λ such that $|\lambda| \leqq 1$ and $f(xy)=\lambda p(x)p(y)$. Also there exist g and h in S (depending on x and y respectively) satisfying $p(x)=g(x)$ and $p(y)=h(y)$. If $g_1=\lambda g$ then $g_1 \in S$. Thus $f(xy)=g_1(x)h(y) \in S(x)S(y)$. Hence $S(xy) \subset S(x)S(y)$ for all $x, y \in E$ and S is collectionwise multiplicative.

(ii) Sufficiency is clear.

Suppose S° is right A-convex. For $y \in E$ there is $\lambda_y > 0$ such that $S^\circ y \subset \lambda_y S^\circ$. If p is as in the proof of (i) above then p satisfies all other properties except that submultiplicativity is replaced by $p(xy) \leqq \lambda_y p(x)$ for all $x, y \in E$. So $|f(xy)| \leqq p(xy) \leqq \lambda_y p(x)$. Therefore, $f(xy) = \mu\lambda_y p(x)$ for some μ with $|\mu| \leqq 1$. Let $g_2 = \mu g$, where $g \in S$ is such that $p(x) = g(x)$. Then $f(xy) = \lambda_y g_2(x)$. So $S(xy) \subset \lambda_y S(x)$ for all $x, y \in E$. Hence S is collectionwise right multiplicative-translation invariant. Similarly we can prove for other parts.

(2.4) THEOREM. *There exists a locally m-convex topology on E compatible with duality (E, E') if and only if there exists a family $\mathscr{S}$ of absolutely convex, $w(E', E)$-compact, collectionwise multiplicative sets in E' that cover E'.*

(2.5) COROLLARY. *The Mackey topology $\tau(E, E') = \chi(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact set is contained in some absolutely convex, $w(E', E)$-compact, collectionwise multiplicative set in E'.*

(2.6) THEOREM. *There exists a (right, left) A-convex topology on E compatible with duality (E, E') if and only if there is a family $\mathscr{S}$ of absolutely convex, $w(E', E)$-compact, collectionwise (right, left) multiplicative-translation invariant sets in E' that cover E'.*

(2.7) COROLLARY. *$\tau(E, E') = \Sigma(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact subset of E' is contained in some absolutely convex, $w(E', E)$-compact, collectionwise multiplicative-translation invariant set.*

(2.8) REMARK. Since the existence of $\chi(E, E')$ $(\Sigma(E, E'))$ is equivalent to the existence of some locally m-convex (A-convex) topology on E compatible with (E, E'), Theorems (2.4) and (2.6) give an answer to question (3.7) in [**4**].

(2.9) REMARK. If there are both A-convex and locally m-convex topologies on E compatible with (E, E') then $\chi(E, E') = \Sigma(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact, collectionwise multiplicative-translation invariant set in E' is contained in an absolutely convex, $w(E', E)$-compact, collectionwise multiplicative set in E'. This gives a partial answer to problem (3.6) in [**4**].

(2.10) EXAMPLE. Let E be the algebra of complex (or real) polynomials without constant term and E' be the subspace of E^* generated by $\{g_i: i=1, 2, \cdots\}$, where $g_i(e_j) = \delta_{ij}$, $e_j(x) = x^j$ for $i, j = 1, 2, \cdots$. Then $(E, w(E, E'))$ is a locally m-convex algebra having no nonzero continuous multiplicative linear functionals (see Proposition 3 and discussion thereafter in [**8**]). By Theorem (2.4) there is a family $\mathscr{S}$ of absolutely convex,

$w(E', E)$-compact, collectionwise multiplicative sets in E' that cover E'. In fact, if $G_n = \{ng_i : 1 \leqq i \leqq n\}$, then its absolutely convex, $w(E', E)$-closed hull H_n in E' is $w(E', E)$-compact. Also the polar G_n° of G_n in E is idempotent and $H_n^\circ = G_n^\circ$. So by Lemma (2.3), H_n is collectionwise multiplicative.

This example shows that a collectionwise multiplicative set need not contain even a single nonzero multiplicative linear functional.

(2.11) EXAMPLE. Let E be the algebra $\boldsymbol{m}$ of bounded complex (or real) sequences with pointwise addition and multiplication and let E' be the space l_1 of absolutely summable sequences. Then the Mackey topology $\tau(E, E')$ is the same as the strict topology β on E considered as the space $C_b(S)$ of bounded continuous complex (or real) functions on the space S of positive integers with the discrete topology ([**2**], [**3**], and [**4**]).Let κ be the compact open topology on E. By Corollary (3.3) in [**4**], there is no locally m-convex topology on E between β and κ. The dual of (E, κ) is the space of sequences with only a finite number of nonzero elements and therefore $\kappa < w(E, E')$.

(i) E is not locally m-convex under any topology compatible with (E, E'). So there exists no family of absolutely convex, $w(l_1, \boldsymbol{m})$-compact (and therefore, $\|\cdot\|_1$-compact), collectionwise multiplicative sets that cover l_1.

(ii) (E, β) has the Mackey topology and is A-convex [**4**]. So every absolutely convex, $w(l_1, \boldsymbol{m})$-compact subset of l_1 is contained in an absolutely convex, $w(l_1, \boldsymbol{m})$-compact, collectionwise multiplicative-translation invariant set.

3. E with the weak topology $w(E, E')$.

In this section E will denote the space E with the weak topology $w(E, E')$. For $B \subset E$ let E_B denote the linear hull of B.

(3.1) LEMMA. *Suppose that E has hypocontinuous multiplication. Let g be in E' and B be an absolutely convex bounded subset of E. Then the kernel $K(g)$ of g contains a closed subspace J of finite codimension in E such that $K(g)$ contains JE_B and E_BJ.*

PROOF. Let V be the polar of $\{g\}$ in E. Since the multiplication in E is hypocontinuous there exists a finite set $F = \{f_i : 1 \leqq i \leqq n\}$ such that $V \supset (BF^\circ) \cup (F^\circ B)$. Let $J = \{x \in E : f_i(x) = 0,\ 1 \leqq i \leqq n\}$. Then $JB \subset F^\circ B \subset V$ and also J is a closed subspace of finite codimension in E. Also $JE_B = JB \subset V = \{g\}^\circ$ and as JE_B is a linear space $JE_B \subset K(g)$. Similarly $E_BJ \subset K(g)$.

(3.2) THEOREM. *If E is the closed linear hull of a bounded subset of itself and E has hypocontinuous multiplication then E has jointly continuous multiplication.*

PROOF. Let B be an absolutely convex bounded subset of E such that $E=E_B^{-}$, where '$^{-}$' denotes the closure in E. Let g be in E'. Let J be as in the proof of the above lemma. Then $JE=JE_B^{-}\subset(JE_B)^{-}\subset(K(g))^{-}=K(g)$. Similarly, $EJ\subset K(g)$. Theorem 2 of Warner [**8**] now gives that E has jointly continuous multiplication.

(3.3) COROLLARY. *If E is the closed linear hull of a bounded set then E is locally m-convex if and only if E is A-convex if and only if it has jointly continuous multiplication if and only if it has hypocontinuous multiplication.*

PROOF. Combine Theorem (3.4) in [**3**], Theorem 1 in [**9**] and Theorem (3.2) above.

(3.4) REMARK. If a locally convex Hausdorff space is the closed linear hull of a bounded set i.e. it is *boundedly generated* (in short, BG) in the terminology of [**6**] then it is BG under each topology compatible with duality (Remark 10 in [**1**]). Every normed linear space is BG and a product of BG spaces is again BG [**6**] (see also Remark 10 in [**1**] and [**2**]). Thus our results are applicable to a large class of algebras.

(3.5) EXAMPLE. The algebra $(\boldsymbol{m}, w(\boldsymbol{m}, l_1))$ is BG but not locally m-convex ([**2**], and Example (2.11) (i) above). So it is not A-convex and does not have hypocontinuous multiplication.

(3.6) EXAMPLE. Let E be the algebra of all complex (or real) continuous functions on the interval $[0, 1]$ with pointwise addition and multiplication equipped with the weak topology resulting from the sup norm topology. Then E is a BG space. Warner [**8**] has shown that E does not have jointly continuous multiplication. Therefore, E is not A-convex and E does not have hypocontinuous multiplication. Thus the claim made in the second part of Examples 3.12 in [**5**] is not valid.

(3.7) EXAMPLE. Consider the algebra φ of complex (or real) sequences with only a finite number of nonzero elements. Then its algebraic dual is the space ω of all complex (or real) sequences under the duality given by $f(x)=\sum_{n=1}^{\infty}\xi_n\zeta_n$ for $x=(\xi_n)\in\varphi$ and $f=(\zeta_n)\in\omega$. So the Mackey topology $\tau(\varphi,\omega)$ is the finest locally convex topology on φ and therefore is the same as the direct sum topology. Also bounded sets are finite-dimensional and every absolutely convex absorbent set is a neighborhood of o in φ. Moreover, ω is the α-dual of φ and $\tau(\varphi,\omega)$ is the same as the normal topology, a base of neighbourhoods of o which is given by

$$\left\{U_f=\left\{x=(\xi_n)\in\varphi:\sum_{n=1}^{\infty}|\xi_n\zeta_n|\leqq 1\right\}, f=(\zeta_n)\in\omega\right\}\qquad [7, §30.1].$$

Let $V_f=\{x\in\varphi:\sum_{n=1}^{\infty}|\xi_n\zeta_n|\leqq 1,\ \sum_{n=1}^{\infty}|\xi_n\eta_n\zeta_n|\leqq\sum_{n=1}^{\infty}|\eta_n\zeta_n|$ for all $y=(\eta_n)\in\varphi\}$. Then $V_fV_f\subset V_f\subset U_f$ and also V_f is an absolutely convex absorbent set and thus a neighbourhood of o in $\tau(\varphi, \omega)$. So $\tau(\varphi, \omega)$ is locally m-convex.

Now let E denote the space φ with the weak topology $w(\varphi, \omega)$. Then E has hypocontinuous multiplication but does not have jointly continuous multiplication.

If B is bounded on E then there exists an integer N and an $\alpha \geqq 0$ such that $B \subset \{x=(\xi_n): \xi_n=0 \text{ for } n>N \text{ and } |\xi_n| \leqq \alpha \text{ for } n \leqq N\}$. Let $f=(\zeta_n) \in E'=\omega$ and let U be its polar in E. For $n \leqq N$, let $g_n \in E'$ be given by $g_n(x)= N\alpha|\zeta_n|\xi_n$, $x=(\xi_n) \in E$. Then the polar V of $\{g_n: 1 \leqq n \leqq N\}$ is a neighbourhood of o in E. Also $VB \subset U$. Thus E has hypocontinuous multiplication.

Now consider $f=(\zeta_n) \in E'$ given by $\zeta_n=1$ for all n. If E is locally m-convex then by Theorem 1 of [**8**], the kernel $K(f)$ of f contains an ideal J of finite codimension. Let $x\ (\neq 0) \in J$. Let $y=(\eta_n) \in E$ be given by $\eta_n=\bar{\xi}_n$ $(n=1, 2, \cdots)$. Then $xy \in J$. Now $f(xy)=\sum_{n=1}^{\infty} |\xi_n|^2 \neq 0$. So $xy \notin K(f)$, which gives a contradiction. So E is not locally m-convex and is, therefore, not A-convex and does not have jointly continuous multiplication.

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