PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

A NOTE ON ZERO-DIMENSIONAL SPACES WITH THE STAR-FINITE PROPERTY

HANS-CHRISTIAN REICHEL

ABSTRACT. The paper provides necessary and sufficient conditions for a weakly zero-dimensional metrizable space to be strongly paracompact, i.e., to have the star-finite property. The characterizations use special basis properties of uniformities which induce the topology of X, and yield further characteristics of the class of all metric spaces with ind $X=0$ and Ind $X>0$.

1. **Introduction.** A space X is said to have the *star-finite property*, if every open covering is refined by a star-finite open covering; i.e. any member of the refinement meets at most a finite number of other elements of the refinement. Spaces with the star-finite property are often called *strongly paracompact*. (As it is known, for locally compact spaces this property coincides with paracompactness.)

In dimension theory the star-finite property plays an important role. P. Roy [Bull. Amer. Math. Soc. **68** (1962), 609–613] has constructed a weakly zero-dimensional metric space X, i.e. ind $X=0$, which certainly does not have the star-finite property. The purpose of this paper is to characterize all weakly zero-dimensional metrizable spaces which have the star-finite property. Using results of (among others) B. Banaschewski, B. Fitzpatrick and R. M. Ford, K. Morita, J. de Groot, A. F. Monna and P. Nyikos this yields nice characterizations of the class of all metric spaces X with ind $X=0$ and Ind $X>0$.

The conditions of these characterizations involve special basis properties and are outside the realm of dimension theory. At the same time, characterizations of all *order-totally-paracompact* [**2**] metrizable spaces with ind $X=0$ are obtained.

2. **Preliminaries.** A space X has Ind $X=0$ (ind $X=0$), if every pair of disjoint closed subsets (points) can be separated by open-closed sets. A. F. Monna and B. Banaschewski have shown that the topology of every space X with ind $X=0$ can be induced by a uniformity $\mathscr{U}$ on X which has a base $\{U_i | i \in I\}$ of equivalence relations, i.e. $U_i \circ U_i = U_i$, $i \in I$ ([**4**], [**1**]). (As a defining family of pseudometrics for $\mathscr{U}$ take e.g.

Received by the editors April 13, 1972 and, in revised form, February 10, 1973.
AMS (MOS) subject classifications (1970). Primary 54D20, 54E15; Secondary 54F45.

$\{d_A | d_A(x,y)=1$ iff $x \in A$, $y \notin A$ or vice versa; moreover $d_A(x,y)=0$ in all other cases; A is a clopen subset of $X\}$.)

If such a space has a *countable* base $\{U_i | i=1,2,\cdots\}$ of equivalence relations, there is a *nonarchimedean metric* d on X, which is compatible with $\mathscr{U}$; in other words, d satisfies the strong triangular inequality:

$$d(x,y) \leqq \max(d(x,z), d(z,y)) \quad \text{for all } x, y, z \in X.$$

Clearly, if $d(x,z)=\varepsilon$, $d(z,y)=\eta$, $\eta \leqq \varepsilon$, then $(x,z) \in U_\varepsilon \in \mathscr{U}$, $(z,y) \in U_\eta \subset U_\varepsilon$. Consequently, $(x,y) \in U_\varepsilon^2 = U_\varepsilon$ which is equivalent to $d(x,z) \leqq \max(d(x,z), d(z,y))$.

Now let X be a nonarchimedeanly metrizable space and $B_\varepsilon(y) = \{(x,y) | d(x,y) < \varepsilon\}$, then the strong triangular inequality implies $B_1 \subset B_2$ or $B_1 \supset B_2$ whenever $B_1 \cap B_2 \neq \varnothing$. It follows that any such space (X,d) has the star-finite property [2] and Ind $X=0$ [7]. Conversely, J. de Groot has proved that the topology of any metrizable space X with Ind $X=0$ can be induced by a nonarchimedean metric [3]. (In a quite different setting this theorem has been proved by F. Hausdorff before.)

Now we can prove the following theorem.

3. The results.

THEOREM 1. *A weakly zero-dimensional metrizable space X has the star-finite property if and only if the topology of X can be induced by a uniform structure which has a totally ordered base (ordered by inclusion) of equivalence relations.*

PROOF. By a theorem of K. Morita [5] the star-finite property guarantees ind $X=$ Ind X for every metrizable space X. Thus any such metrizable space X, ind $X=0$, has Ind $X=0$ and therefore is nonarchimedeanly metrizable. The induced uniform structure $\mathscr{U}$ obviously has a totally ordered base of equivalence relations:

$$\{U_\varepsilon = \{(x,y) \mid d(x,y) < \varepsilon\}, \varepsilon > 0\}.$$

Conversely, let $\mathscr{B} = \{U_i | i \in I\}$ be a totally ordered base of equivalence relations for the uniform space $(X, \mathscr{U})$; then every U_i induces a partition of X, and, of course, the clopen sets of all these partitions form a base for the topology. Moreover, if two members A, B of this base have nonempty intersection, they belong to different partitions one of which refines the other. So we essentially obtain a base with the following property: two basis sets either have empty intersection or one contains the other. And this implies the star-finite property [2].

REMARK. The second part of the proof goes through without any metrizability condition. We can construct a type of universal space for arbitrary uniform spaces X having a totally ordered base $\mathscr{B}$ of equivalence relations: Let $\{U_\sigma|\sigma<\tau\}$ be a well-ordered cofinal subset of $\mathscr{B}$, then every U_i induces a partition of X into $\mathfrak{m}_i$ clopen sets. Now let $\alpha=\sup \mathfrak{m}_i$ and let A be an arbitrary set of cardinality α. For each equivalence class $B_\sigma(x)$, $x \in X$, we arbitrarily associate an element of A, associating distinct elements with distinct classes. This procedure yields an inverse system of sets C_σ and we can build the projective limit $C=\text{proj lim}\{C_\sigma|\sigma<\tau\}$ in the category of sets. The points of C can be identified with transfinite sequences $(a_\sigma|\sigma<\tau) \in A^\tau$. Now for any sequence (a_σ) define the system

$$\{B_\mu(a_\sigma) = \{(b_\sigma) \mid a_\sigma = b_\sigma, \forall \sigma < \mu\}, \mu < \tau\}$$

to be a local base for a topology t on A^τ. Thus C has the topology inherited from the space (A^τ, t) and, obviously, X is homeomorphically imbeddable into C.

Spaces A^τ with the topology considered above have been studied to some extent by A. K. Steiner and E. F. Steiner under the name *The natural topology on spaces* A^τ [J. Math. Anal. Appl. **19** (1967), 174–178].

P. Roy [§1] was the first to construct a metric space X with $\operatorname{ind} X=0$ and $\operatorname{Ind} X\neq 0$, thus solving the problem of the equivalence of "ind" and "Ind" for metric spaces in the negative. By Theorem 1 we obtain a characterization of all such spaces:

COROLLARY 2. *A metrizable space X has* $\operatorname{ind} X=0$ *and* $\operatorname{Ind} X>0$ *if and only if no uniformity $\mathscr{U}$ compatible with the topology of X has a totally ordered base.*

PROOF. By Theorem 1 we only have to prove the sufficiency of the condition; but this is obvious by the theorem of Hausdorff-de Groot [§2].

4. **Order-totally-paracompact spaces.** R. M. Ford in his thesis defined the concept of total paracompactness and showed that $\operatorname{ind} X=\operatorname{Ind} X$ for totally paracompact metric spaces. Later on, Fitzpatrick and Ford [2] generalized this concept to the concept of order-total-paracompactness which again guarantees $\operatorname{ind} X=\operatorname{Ind} X$ for metric spaces X. Order-total-paracompactness generalizes especially the star-finite property, which is not induced by total paracompactness alone. (Compare also J. A. French in [Duke Math. J. **38** (1971), 251–253].) Let us recall the definitions: X is *totally paracompact* iff every basis for X has a locally finite subcollection covering X. And X is *order-totally-paracompact* if for any base G there is a linearly ordered collection $(H, <)$ of open sets covering X such that:

(i) every $h \in H$ is contained in an element $g \in G$ such that $B(h)\subset B(g)$, where "B" denotes the boundary of the set.

(ii) H is initially locally finite, that is: for any $h \in H$ the collection of all $h_i < h$ ($h_i \in H$) is locally finite at each point of $\bar{h}$.

The importance of condition (i) is showed by a result of H. Tamano, *Note on paracompactness* [J. Math. Kyoto Univ. **3** (1963), 137–143]: a regular space X is paracompact iff for every open cover G of X there is a linearly ordered collection $(H, <)$ of open sets refining G such that H is initially locally finite. Other interesting references in this connection are: H. Tamano and J. E. Vaughan, *Paracompactness and elastic spaces* [Proc. Amer. Math. Soc. **28** (1971), 299–303]; J. E. Vaughan, *Linearly ordered collections and paracompactness* [Proc. Amer. Math. Soc. **24** (1970), 186–192] and Y. Katuta, *A theorem on paracompactness of product spaces* [Proc. Japan. Acad. **43** (1967), 614–618].

Now, by Theorem 1, we get the following characterization:

THEOREM 3. *For metric spaces X the following properties are equivalent*:

(i) ind $X=0$ *and X has the star-finite property.*

(ii) ind $X=0$ *and X is order-totally-paracompact.*

(iii) Ind $X=0$.

(iv) *The topology of X can be induced by a uniformity $\mathcal{U}$ which has a* ***countable*** *base consisting of equivalence relations U_i on X.*

(v) *The topology of X can be induced by a uniformity $\mathcal{U}$ which has a* ***totally ordered*** *base consisting of equivalence relations U_i on X.*

PROOF. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) has been proved by Fitzpatrick and Ford in [**2**]. (iii)$\Rightarrow$(iv) follows by the theorem of Hausdorff-de Groot (compare also [**1**]) and analogously (iv)$\Rightarrow$(v) because of the metrizability of X. The implication (v)$\Rightarrow$(i), and thus the equivalence of all these properties, is proved in our Theorem 1.

REMARK. The negation of any of these properties gives interesting characterizations of the so-called Roy spaces, i.e. the class of all metric spaces X with ind $X=0$ and Ind $X>0$.

ACKNOWLEDGEMENT. The author would like to thank the referee and P. Nyikos for interesting and worthwhile communications.

REFERENCES

1. B. Banaschewski, *Über nulldimensionale Räume*, Math. Nachr. **13** (1955), 129–140. MR **19**, 157.

2. B. Fitzpatrick, Jr. and R. M. Ford, *On the equivalence of small and large inductive dimension in certain metric spaces*, Duke Math. J. **34** (1967), 33–37. MR **34** #5059.

3. J. de Groot, *Non-archimedean metrics in topology*, Proc. Amer. Math. Soc. **7** (1956), 948–953. MR **18**, 325.

4. A. F. Monna, *Remarques sur les métriques non-archimédiennes.* I, II, Nederl. Akad. Wetensch. Proc. **53**, 470–481=Indag. Math. **12** (1950), 122–133; ibid. **53**, 625–635=Indag. Math. **12** (1950), 179–191. MR **12,** 41, 1002.

5. Jun-iti Nagata, *Modern dimension theory*, Bibliotheca Mathematica. Vol. VI, Interscience, New York, 1965. MR **34** #8380.

6. P. Nyikos, *N-compact spaces*, Dissertation, Carnegie-Mellon University, Pittsburgh, Pa., 1972.

7. P. Nyikos and H. C. Reichel, *On the structure of zero-dimensional spaces* (to appear).

8. H. C. Reichel, *On the metrization of the natural topology on the space A^B*, J. Math. Anal. Appl. **34** (1971), 644–647. MR **43** #1116.

9. P. Ucsnay, *Über uniforme Räume*, Bonn. Math. Schr. **48** (1970), 21–30.

10. A. C. M. van Rooij, *Non-archimedean uniformities*, Kyungpook Math. J. **10** (1970), 21–30.

Mathematisches Institut, Universität Wien, Vienna, Austria