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CATEGORIES OF H-SPACES

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ABSTRACT. Fuchs' [1] category of H-spaces and homotopy classes of shm-maps is shown equivalent to a simpler category of fractions.

Introduction. When algebra and topology are combined, one often encounters difficulties not otherwise present. For example a continuous homomorphism between two topological monoids, $f: G \to H$, may be a homotopy equivalence of their underlying topological spaces, but the homotopy inverse need not be a homomorphism. One solution to this problem is given by Fuchs in [**1**] by enlarging the class of allowable maps between monoids, and then passing to homotopy classes. This note shows that Fuchs' procedure is equivalent to the more immediate categorical process of passing to an appropriate category of fractions.

For later results in his paper, Fuchs assumes that all spaces have the homotopy type of CW-complexes. For the purposes of this note it suffices to have compactly generated spaces, so that the exponential law works.

Let $\mathcal{M}on$ be the category of topological monoids and continuous homomorphisms, and $\mathcal{H}$ Fuchs' category of topological monoids and homotopy classes of H-homomorphisms. Then one has the following result.

THEOREM. $\mathrm{Ho}(\mathcal{M}on)=\mathcal{H}$, *where* $\mathrm{Ho}(\mathcal{M}on)$ *is the category of fractions of* $\mathcal{M}on$ *with respect to the class of maps which are homotopy equivalences of the underlying topological spaces, i.e. the category obtained by adding formal inverses for all such maps.*

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1. The universal monoid.

DEFINITION 1. Let G be a topological monoid.

$$UG = \coprod G^n \times I^{n-1}/\text{equivalence relation}$$

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where the equivalence relation is generated by equations of the form

(a) $(g_1, g_2, \cdots, g_n, t_1, t_2, \cdots, t_{n-1}) = (g_1, \cdots, g_i g_{i-1}, \cdots, g_n, t_1, \cdots, \hat{t}_i, \cdots, t_{n-1})$ if $t_i = 1$.

(b) $(e, g_1, \cdots, g_{n-1}, t_1, t_2, \cdots, t_{n-1}) = (g_1, g_2, \cdots, g_{n-1}, t_2, \cdots, t_{n-1})$ and $(g_1, \cdots, g_{n-1}, e, t_1, \cdots, t_{n-1}) = (g_1, \cdots, g_{n-1}, t_1, \cdots, t_{n-2})$ where e is the identity of G.

The coproduct and equivalence relations are taken in the category of topological spaces. I is the unit interval $[0, 1]$. A multiplication is defined on UG by

$$[(g_1, \cdots, g_n, t_1, \cdots, t_{n-1})][h_1, \cdots, h_n, s_1, \cdots, s_{n-1}] = [g_1, \cdots, g_n, h_1, \cdots, h_n, t_1, \cdots, t_{n-1}, 0, s_1, \cdots, s_{n-1}].$$

It is easy to check that $[e]$ is an identity for this (clearly) associative multiplication. If $f: G \to G'$ is a homomorphism, define Uf to be

$$Uf = \coprod f^n \times I^{n-1}/\text{e.r.}$$

which is well defined since f is a homomorphism, and clearly multiplicative. So $U: \mathcal{M}on \to \mathcal{M}on$ is a functor.

REMARK. U has additional structure which makes it a highly homotopy coassociative cotriple [2]. In a later paper [3] all of this higher structure will be examined, but for purposes of the present note, it suffices to have simply the homotopy cotriple structure used below.

It is immediate that a homomorphism from UG to G' is the same thing as what Fuchs calls a regular H-homomorphism [**1**, §6]. Two homomorphisms $f, f': UG \to G'$ and a homotopy $H: UG \times I \to G'$ such that $H(\ , t)$ is a homomorphism for all t constitute a homotopy between H-homomorphisms [**1**, §3].

Composition of H-homomorphisms can be described by the methods of Kleisli [**4**] using the following natural transformation: Define $\delta: U \to U^2$ by

$$\delta([g_1, \cdots, g_n, t_1, \cdots, t_{n-1}]) = [[\mathcal{g}_{i_1}], \cdots, [\mathcal{g}_{i_{k+1}}], 2t_{i_1}, \cdots, 2t_{i_k}]$$

where $0 \leqq t_{i_j} \leqq \frac{1}{2}$, $j = 1, \cdots, k$, and

$$[\mathcal{g}_{i_j}] = [g_{i_{j-1}+1}, \cdots, g_{i_j}, 2t_{i_{j-1}+1} - 1, \cdots, 2t_{i_j-1} - 1] \in UG.$$

The t_{i_k} are all those elements of I between 0 and $\frac{1}{2}$ inclusive. I.e., t_{i_1} is the first t_j with $0 \leqq t_j \leqq \frac{1}{2}$ and $i_1 = j$. The other elements of I are counted around the t_i, e.g., t_{i_2+k}, and lie in $[\frac{1}{2}, 1]$.

Since δ has no effect on the G component, except possibly to multiply some elements together, δ is natural with respect to homomorphisms. It is well defined at points where some $t_i = \frac{1}{2}$ by virtue of relation (a) of Definition 1, and the definition of multiplication in UG.

EXAMPLE. $\delta([g_1, g_2, g_3, g_4, 0.3, 0.4, 0.6]) = [[g_1], [g_2], [g_3, g_4, 0.2], 0.6, 0.8]$.

Composition of two H-homomorphisms $f: UG \to G'$, $k: UG' \to G''$ can now be defined by

$$UG \xrightarrow[\delta]{} U^2G \xrightarrow[Uf]{} UG' \xrightarrow[k]{} G''. \tag{1}$$

This is the same as the composition [**1**, 2.2(c)]. It follows from [**1**, 3.4] that the category with objects topological monoids and maps from G to G' homotopy classes of homomorphisms from UG to G' and composition given by (1) is $\mathscr{H}$. (*Note.* The homotopy classes of homomorphisms must be with respect to homotopy through homomorphisms. See remark following Definition 1.)

The final piece of structure which will be needed is another natural transformation $p: U \to \mathcal{M}on$ defined by

$$p_G([g_1, \cdots, g_n, t_1, \cdots, t_{n-1}]) = g_1 g_1 \cdots g_n.$$

Note that p_G is a homotopy equivalence of the underlying topological spaces: By shrinking the t's any point in UG is homotopy equivalent to $g_1 \cdots g_n$. Moreover, $[p_G]$ is the identity of $\mathscr{H}$.

2. **The equivalence.** In this section $\mathrm{Ho}(\mathcal{M}on)$ will mean the category of fractions of $\mathcal{M}on$ with respect to the class of maps which as maps of topological spaces are homotopy equivalences. It can be obtained from $\mathcal{M}on$ by adding one copy of every such map, taking the free category and then factoring out the relations already given by the category structure in $\mathcal{M}on$ and the relations making each copy of a map in the class an inverse for its original. A general map from X_1 to X_4 can be written (not uniquely) as a sequence:

$$\begin{tikzcd}
 & Y_1 & & Y_2 & & Y_3 & \\
X_1 \arrow[ur, "f_1"'] & & X_2 \arrow[ul, "g_1"'] \arrow[ur, "f_2"'] & & X_3 \arrow[ul, "g_2"'] \arrow[ur, "f_3"'] & & X_4 \arrow[ul, "g_3"']
\end{tikzcd}$$

where the f_i are homomorphisms and the g_i are homotopy equivalences and homomorphisms.

There is a functor $\mathrm{Ho}(\mathcal{M}on) \xleftarrow{\gamma} \mathcal{M}on$ which is the identity on objects and takes a map f into its class in the fraction category (the analogue of the map $Z \to Q$ which sends $n \to n/1$). Any functor which sends homotopy equivalences into isomorphism factors through γ. (The fraction category could also be defined in terms of this universal property. For generalities on fraction categories see [**5**], [**6**].)

Define $I:\mathscr{M}on\to\mathscr{H}$ as follows: $IG=G$, $I(f)=[p_G\cdot f]:UG\to H$, for $f:G\to H$.

It must be shown that I is a functor. $I(\mathrm{id}_G)=[p_G]$ which is the identity in $\mathscr{H}$. Let $f:G\to H$, $g:H\to L$. Then

$$\begin{aligned} I(f\cdot g) &= [p_G\cdot f\cdot g] \\ &= [Uf\cdot p_H\cdot g] \quad \text{by naturality of } p, p_G\cdot f = Uf\cdot p_H \\ &= [\delta_G\cdot Up_G\cdot Uf\cdot p_H\cdot g] \\ &\qquad\qquad \text{since } \delta_G\cdot Up_G \text{ is homotopic to the identity} \\ &= [\delta_G\cdot U(p_G\cdot f)\cdot(p_H\cdot g)] = I(f)\cdot I(g). \end{aligned}$$

It follows from [**1**, Satz in §4] that if f is a homotopy equivalence, $I(f)$ is an isomorphism in $\mathscr{H}$. So I factors through γ, and the corresponding functor from $\mathscr{H}o(\mathscr{M}on)$ to $\mathscr{H}$, will be denoted $\mathscr{I}$.

Now consider the functor $J:\mathscr{H}\to\mathrm{Ho}(\mathscr{M}on)$ defined by $JG=G$. For $f:UG\to H$ in $[f]$ in $\mathscr{H}$, $J([f])=\gamma(p_G)^{-1}\cdot\gamma(f)$. J is well defined if $f\simeq g$ through homomorphisms implies $\gamma(f)=\gamma(g)$. But $f\simeq g$ means there is an M where (α) $M:UG\times I\to H$ such that if $i_\varepsilon:UG\to UG\times I:x\to(x,\varepsilon)$, $\varepsilon=0,1$, (β) $i_0M=f$ and $i_1M=g$. Let $UG\otimes I$ be the free topological monoid on $UG\times I$ modulo the relations $(x,t)(x',t)=(xx',t)$. Then since M is a homotopy through homomorphisms, it determines a map $M':UG\otimes I\to H$ such that (α) and (β) hold with M replaced by M' and $UG\otimes I$ replacing $UG\times I$. i_0 and i_1 are homomorphisms by virtue of the relations. Finally $r:UG\otimes I\to UG$ given by $r([x,t])=x$ is a homomorphism, and a homotopy inverse for both i_0 and i_1. So $\gamma(r)\cdot\gamma(i_\varepsilon)=\mathrm{id}_{UG}$ for $\varepsilon=0,1$. Consequently $\gamma(r)\cdot\gamma(f)=\gamma(r)\cdot\gamma(g)$, by β, and, since $\gamma(r)$ is an isomorphism, $\gamma(f)=\gamma(g)$.

So J is well defined. It must be shown that $J([f][g])=J([f])J([g])$. $f:UG\to Hg:UH\to L$

$$\begin{aligned} J([f][g]) &= J([\delta_G\cdot Uf\cdot g]) \\ &= \gamma(p_G)^{-1}\cdot\gamma(\delta_G)\cdot\gamma(Uf)\cdot\gamma(g) \quad \text{by definition} \\ &= \gamma(p_G)^{-1}\cdot\gamma(\delta_G)\cdot\gamma(Up_G)\cdot\gamma(f)\cdot\gamma(p_H)^{-1}\cdot\gamma(g) \\ &\qquad\qquad \text{by naturality, see Figure 1} \\ &= \gamma(p_G)^{-1}\cdot\gamma(f)\cdot\gamma(p_H)^{-1}\cdot\gamma(g) \\ &\qquad\qquad \text{since } p \text{ is a homotopy counit for } \delta \\ &= J([f])J([g]). \end{aligned}$$

$$\begin{tikzcd} UG \arrow[r, "\delta_G"] \arrow[d, "p_G"'] \arrow[dr, equal] & U^2G \arrow[r, "Uf"] \arrow[d, "Up_G"'] & UH \arrow[r, "g"] \arrow[d, "p_H"'] & L \\ G \arrow[r] & UG \arrow[r, "f"'] & H & \end{tikzcd}$$

FIGURE 1

Since the identity for $\mathscr{H}$ is $[p_G]$, it is immediate that $J(G)=\mathrm{id}_G$. So it remains to show J and $\mathscr{I}$ are inverses. There is no problem with objects, so it suffices to look at maps. Since $\mathscr{I}$ is a functor, it is also sufficient to look at $J\mathscr{I}(\gamma(f))$ for f a map in $\mathscr{M}on$. But $J\mathscr{I}(\gamma(f))=JI(f)=J([p_G\cdot f])=\gamma(p_G)^{-1}\cdot \gamma(p_G)\cdot\gamma(f)=\gamma(f)$. $\mathscr{I}J([f])=\mathscr{I}(\gamma(p_G)^{-1}\cdot\gamma(f))=I(p_G)^{-1}\cdot I(f)$. Recall that in $\mathscr{H}$, f is given by a map $UG\to H$, so the composition is well defined. $I(p_G)^{-1}\cdot I(f)=[f]$ by the definition of $I(p_G)^{-1}$: apply [ ] to Figure 2, and recall $[p_G]$ is the identity. Square Δ commutes by definition of $I(p_G)^{-1}$.

$$\begin{tikzcd}
UG \arrow[r, "\delta_G"'] \arrow[drr, "p_G"'] & U^2G \arrow[r, "U(I(p_G)^{-1})"'] & U^2G \arrow[r, "p_{UG}"'] & UG \arrow[r, "f"'] \arrow[dl, "p_G"] \arrow[dl, phantom, "\Delta", shift right=3ex] & H \\
& & G & &
\end{tikzcd}$$

FIGURE 2

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