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UNIQUE FACTORIZATION IN GRADED POWER SERIES RINGS

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ABSTRACT. It is shown that the graded ring $R[x_1, x_2, \cdots][[t]]$ of homogeneous power series is a graded UFD if R is a regular UFD, the degrees of the x_i are positive and tend to ∞, and t has degree -1. In particular this applies to $MU^*(CP^\infty)$ and $BP^*(CP^\infty)$.

1. **Introduction.** It is well known that if R is a unique factorization domain (UFD) then so is any polynomial ring over R; on the other hand, P. Samuel [5] has shown that the power series ring $R[[t]]$ need not be a UFD. In the positive direction, if R is a regular UFD then so is $R[[t]]$; in particular if R is a principal ideal domain then $R[[x_1, \cdots, x_n]]$ is a UFD (see e.g. [1], [4] or [5]). At present it is apparently not known if $R[x_1, x_2, \cdots][[t]]$ is a UFD, even when R is a field.

In this note we study unique factorization of *homogeneous* power series over *graded* rings. If $S=(S_n)_{n\in Z}$ is a graded ring, we call $s\in S_n$ a homogeneous element of S of degree n, and write $|s|=n$. For a commutative graded ring S, let $S[[t]]$ denote the graded ring of homogeneous power series over S, where $|t|=-1$; thus $S[[t]]_n$ consists of power series $f=\sum_{i=0}^\infty f_i t^i$ with $|f_i|=n+i$. Our main result, Theorem 3.1, is that $S[[t]]$ is a graded UFD in case $S=R[x_1, x_2, \cdots]$ is a graded polynomial ring over a regular UFD R, where the degrees $|x_i|$ are positive and converge to ∞. We note that with this choice of gradings $R[x_1, x_2, \cdots][[t]]$ is isomorphic to the inverse limit of the graded rings $R[x_1, \cdots, x_n][[t]]$; in §2 we show that $R[x_1, \cdots, x_n][[t]]$ is a graded UFD (Corollary 2.3), and then pass to the limit in §3.

In particular, it follows that $MU^*(CP^\infty)$ and $BP^*(CP^\infty)$ are graded UFD's, where $MU^*(\ )$ denotes complex cobordism and $BP^*(\ )$ is Brown-Peterson cohomology. (I apologize for using homology rather than cohomology indexing.) For MU this result has been stated without proof and used by T. tom Dieck (see [2, p. 365] or [3, p. 34]); since the proof is deeper than one may first expect, it seems worthwhile to publish it.

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2. **A reduction of the problem.** Let S be a graded integral domain, and $S[[t]]$, $|t|=-1$, the graded ring of homogeneous power series over S.

(2.1) PROPOSITION. *$S[[t]]$ is a graded UFD if and only if the ring $S[[t]][t^{-1}]_0$ is a UFD.*

Notice that $S[[t]][t^{-1}]_0$ is the ring of Laurent series $f=\sum_{i=n}^{\infty} f_i t^i$ with $f_i \in S_i$ and n any integer. In particular, if the grading on S is nonnegative then this coincides with $S[[t]]_0$, the ring of power series $f=\sum_{i=0}^{\infty} f_i t^i$ over S with $|f_i|=i$.

PROOF. Suppose that $S[[t]]$ is a UFD. Hence there is a set $\{p_\alpha\}$ of prime elements in $S[[t]]$ so that each nonzero $f \in S[[t]]$ has the form

$$f = ut^{n_0} p_{\alpha_1}^{n_1} \cdots p_{\alpha_r}^{n_r} \tag{2.2}$$

where u is a unit and the exponents $n_0, \cdots, n_r$ are uniquely determined. It is immediate that $S[[t]][t^{-1}]$ and $S[[t]][t^{-1}]_0$ are UFD's, with a complete set of prime elements given by $\{p'_\alpha\}$ where $p'_\alpha=p_\alpha t^{|p_\alpha|}$ has degree 0.

Conversely, assume $S[[t]][t^{-1}]_0$ is a UFD with $\{p'_\alpha\}$ a complete set of prime elements. Choose d_α so that $p_\alpha=t^{-d_\alpha}p'_\alpha$ belongs to $S[[t]]$ and has nonzero constant term (note that $d_\alpha=|p_\alpha|$). One checks easily that the p_α are prime elements in $S[[t]]$, and that each of its nonzero homogeneous elements in a product of the form (2.2). Thus $S[[t]]$ is a graded UFD.

(2.3) COROLLARY. *If $S=R[x_1, \cdots, x_n]$ is a graded polynomial ring, where $|x_i|>0$ and R is a regular UFD, then $S[[t]]$ is a graded UFD.*

PROOF. Since S is nonnegatively graded, $S[[t]][t^{-1}]$ coincides with $S[[t]]_0$. In turn we may identify $S[[t]]_0$ with $R[[y_1, \cdots, y_n]]$, where $y_i=x_i t^{|x_i|}$, which is known to be a UFD (see e.g. [4, Theorem 188]).

3. **The main theorem.** In this section we fix a regular UFD R and let S denote the graded polynomial ring $R[x_1, x_2, \cdots]$ where $|x_i|>0$ and $|x_i|\to\infty$.

(3.1) THEOREM. *$R[x_1, x_2, \cdots][[t]]$ is a graded UFD provided that $|x_i|>0$, $|x_i|\to\infty$ and $|t|=-1$.*

PROOF. For convenience we let S_n denote the graded polynomial ring $R[x_1, \cdots, x_n]$. We define homomorphisms of R-algebras $\phi_n: S\to S_n$ and $\phi_n^N: S_N\to S_n$ for $N\geqq n$ by $x_i\mapsto x_i$ if $i\leqq n$ and $x_i\mapsto 0$ if $i>n$.

These induce homomorphisms

$$\phi_n: S[[t]] \to S_n[[t]], \qquad \phi_n^N: S_N[[t]] \to S_n[[t]], \qquad N \geqq n$$

by acting on the coefficients of the power series. In view of our assumption that $|x_i|>0$ and $|x_i|\to\infty$, it is immediate that the ϕ_n define an isomorphism

$$S[[t]] \cong \varprojlim_n S_n[[t]]$$

of graded rings. We know each $S_n[[t]]$ is a graded UFD, and shall show how to pass to the limit.

To show that $S[[t]]$ is a graded UFD, it suffices to verify the following statements:

(3.2) Each nonzero element is a product of irreducible elements.

(3.3) Each irreducible element is prime.

We recall that f is irreducible if $f=gh \Rightarrow g$ or h is a unit; and f is prime if $f|gh \Rightarrow f|g$ or $f|h$. A prime element is always irreducible; the converse holds in a UFD.

It suffices to establish (3.2) and (3.3) for homogeneous power series $f=\sum_{i=0}^{\infty} f_i t^i$ with $f_0 \neq 0$, since t is evidently a prime element.

As to (3.2), notice that, if $f=gh$ where $f=\sum t_i t^i$ $(f_0 \neq 0)$, $g=\sum g_i t^i$ and $h=\sum h_i t^i$, then $f_0=g_0 h_0$. Since $R[x_1, x_2, \cdots]$ is a UFD, we can easily express f as a product of irreducible elements.

Prior to showing that irreducible elements are prime in $S[[t]]$, we verify the statement:

(3.4) If $f \in S[[t]]$ is irreducible, then for large n also $\phi_n(f) \in S_n[[t]]$ is irreducible.

So let $f=\sum f_i t^i$ $(f_0 \neq 0)$ be irreducible and write $f_0=p_1 \cdots p_s$ where the p_i are irreducible in $R[x_1, x_2, \cdots]$. Choose N so that $n>N \Rightarrow |x_n|>|f_0|$; then for $n \geqq N$ also $\phi_n(f)$ has constant coefficient f_0. Thus $\phi_n(f)$ is a product of r_n irreducible elements with $r_n \leqq s$; evidently we have $r_n \leqq r_{n+1} \leqq s$, so if we increase N sufficiently then $n \geqq N \Rightarrow r_n = n$ is independent of n. Hence we may assume that for $n \geqq N$ we have

$$\phi_n(f) = p_1^{(n)} \cdots p_n^{(n)} \tag{3.5}$$

where the $p_i^{(n)}$ are irreducible elements of $S_n[[t]]$.

We next show that it is possible to arrange that

$$\phi_n^{n+1}(p_i^{(n+1)}) = p_i^{(n)}, \qquad n \geqq N. \tag{3.6}$$

To see this, begin with $\phi_N(f)=p_1^{(N)} \cdots p_r^{(N)}$ and continue by induction on n. Thus if we have achieved $\phi_n(f)=p_1^{(n)} \cdots p_r^{(n)}$ we first choose the $p_i^{(n+1)}$ provisionally so that

$$\phi_{n+1}(f) = p_1^{(n+1)} \cdots p_r^{(n+1)}.$$

Applying ϕ_n^{n+1}, we may permute the $p_i^{(n+1)}$ and obtain $\phi_n^{n+1}(p_i^{(n+1)})=u_i p_i^{(n)}$ with u_i a unit. If we replace $p_i^{(n+1)}$ by $p_i^{(n+1)} u_i^{-1}$ for $i<r$, then we have

$$\phi_n^{n+1}(p_i^{(n+1)}) = p_i^{(n)} \quad \text{for } i < r;$$

from $\phi_n(f)=\phi_n^{n+1}(\phi_{n+1}(f))$ we see that also $\phi_n^{n+1}(p_r^{(n+1)})=p_r^{(n)}$. This establishes (3.6).

Thus for $i=1, \cdots, r$ the elements $\{p_i^{(n)}\}_{n\geqq N}$ determine an element of the inverse limit of the $S_n[[t]]$, hence we obtain $p_1, \cdots, p_r$ in $S[[t]]$ so that $\phi_n(p_i)=p_i^{(n)}$ for $n\geqq N$. This gives $\phi_n(f)=\phi_n(p_1\cdots p_r)$ for $n\geqq N$, so $f=p_1\cdots p_r$. No p_i is a unit and f is irreducible, hence $r=1$ and we have verified (3.4).

We are ready for (3.3). Let f be irreducible in $S[[t]]$ and assume $f|gh$. For large n we know $\phi_n(f)$ is prime, hence $\phi_n(f)$ divides $\phi_n(g)$ or $\phi_n(h)$. Thus we may suppose that $\phi_n(f)$ divides $\phi_n(g)$ for all n. We see immediately that f divides g, as desired. So f is a prime element. Q.E.D.

(3.7) Remark. One easily recognizes as irreducible any power series in $R[x_1, x_2, \cdots][[t]]$ whose constant term is an irreducible element of $R[x_1, x_2, \cdots]$. This covers the power series that arise in [2] and [3]. It is easy to show that such an irreducible element is prime, in both the graded and ungraded case; the point is that the analogue of (3.4) is now automatic.

(3.8) Remark. In view of (2.1), we have shown that the power series ring $R[[x_1, x_2, \cdots]]$ over a regular UFD is a UFD if $|x_i|>0$ and $|x_i|\to\infty$. Its elements have the form $f=\sum_{i=0}^{\infty} f_i$ where $f_i \in R[x_1, x_2, \cdots]_i$ is a polynomial of degree i.

(3.9) Remark. One measure of the difficulty in the ungraded case is that we only obtain a monomorphism of $R[x_1, x_2, \cdots][[t]]$ into the inverse limit of the $R[x_1, \cdots, x_n][[t]]$, hence we cannot prove the analogue of 3.4 by our argument.

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