A NOTE ON GROUPS WITH RELATIVELY COMPACT CONJUGACY CLASSES

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ABSTRACT. In a more general form, the following structure theorem is proved. Let G be a locally compact group with small invariant neighborhoods. Then G has relatively compact conjugacy classes if and only if G is a direct product of a vector group V and a group L where L has a compact open normal subgroup K such that L/K has finite conjugacy classes.

The purpose of this note is to prove the following theorem which is a direct generalization of the basic structure theorem for locally compact abelian groups [2, Theorem 24.30].

THEOREM. *Let $\mathscr{B}$ be a subgroup of $\mathfrak{A}(G)$ containing the inner automorphisms. Let $G \in [SIN]_{\mathscr{B}}$. Then $G \in [FC]_{\mathscr{B}}^{-}$ if and only if G contains $\mathscr{B}$-invariant subgroups V, L and K such that V is a vector group, K is compact and open in L, $L/K \in [FC]_{\mathscr{B}}^{-}$, and $G=VL$ is a direct product of V and L.*

First we establish a few definitions and some notation. All groups considered are Hausdorff and locally compact. The group operation is multiplication. A vector group is one which is topologically isomorphic to the additive structure of R^n with $n \geq 0$. The connected component of the identity of a topological group G is denoted G_e. An element of G is said to be compact if the subgroup it generates has compact closure. The group of topological automorphisms of G is $\mathfrak{A}(G)$. If $\mathscr{B}$ is a subgroup of $\mathfrak{A}(G)$ which contains the inner automorphisms, then the $\mathscr{B}$-orbit of $x \in G$ is $\{\beta(x):\beta \in \mathscr{B}\}$. A subset S of G is said to be $\mathscr{B}$-invariant, if $\beta(s) \in S$ for all $s \in S$ and $\beta \in \mathscr{B}$. We are interested in the following classes of groups.

$G \in [FC]_{\mathscr{B}}^{-}$ if the $\mathscr{B}$-orbits of points have compact closures.

$G \in [SIN]_{\mathscr{B}}$ if there is a neighborhood basis of compact $\mathscr{B}$-invariant neighborhoods at the identity.

$G \in [FD]_{\mathscr{B}}^{-}$ if the $\mathscr{B}$-commutator subgroup, which is the closure of the group generated by $\{x^{-1}\beta(x):\beta \in \mathscr{B}, x \in G\}$, is compact.

If $\mathscr{B}$ actually equals the inner automorphism group, then the subscript and prefix $\mathscr{B}$ is omitted. If H is a $\mathscr{B}$-invariant subgroup of G, then the restriction of $\mathscr{B}$ to H is a subgroup of $\mathfrak{A}(H)$ which, by abuse of notation, is again denoted $\mathscr{B}$. Similar remarks apply to quotients formed by $\mathscr{B}$-invariant subgroups.

The proof of the theorem relies on the following results of Grosser and Moskowitz and on the lemma below. Let $\mathscr{B}$ be a subgroup of $\mathfrak{A}(G)$ containing the inner automorphisms; let $G \in [FC]_{\overline{\mathscr{B}}}^{-}$ and let P be the periodic subgroup of G, that is, P is the set of compact elements of G.

(1) P is a closed $\mathscr{B}$-invariant subgroup of G and the sequence $1 \rightarrow P \rightarrow G \rightarrow W \times D \rightarrow 1$ is exact. Here W is a vector group and D is a discrete torsion-free abelian group [3, Theorem 3.16].

(2) If G is compactly generated, then P is compact [3, Theorem 3.20].

(3) If $G \in [FD]^{-}$, then normal vector subgroups split [3, Corollary 4.3].

LEMMA. *Let $\mathscr{B}$ be a subgroup of $\mathfrak{A}(G)$ containing the inner automorphisms and let $G \in [FC]_{\overline{\mathscr{B}}}^{-}$. If the connected component of the identity $G_e = VK$ is a direct product of a nontrivial $\mathscr{B}$-invariant vector subgroup V and a compact group K, then there is a $\mathscr{B}$-invariant subgroup L such that $G = VL$ is a direct product of V and L with L_e compact.*

PROOF. Let P be the set of compact elements of G. We claim the map $\psi: VP/P \rightarrow V/(V \cap P) = V$ defined by $\psi(vP) = v$ is a topological isomorphism. Since V is σ-compact and P is closed, this follows from [2, Theorem 5.33] providing we show that VP is open, hence closed in G. Let H be any compactly generated open subgroup of G. Then $H \in [FC]^{-}$ and so by (2) $H \in [FD]^{-}$. Furthermore, V is a normal vector subgroup of H so that, by (3), $H = VM$ is a direct product of V with a subgroup M. Since $H = VM \supset G_e = VK$ and V contains no compact elements, $M \supset K$. Furthermore, $M/K \cong H/G_e$ is totally disconnected. Thus M contains a compact open subgroup M_1. Since $VM_1 \subset VP$ and VM_1 is an open subset of G, VP is open in G.

By (1) $G/P = WD$ is a direct product of a vector subgroup W with a discrete subgroup D. Let $\pi_1: G \rightarrow G/P$ and $\pi_2: WD \rightarrow W$ be the canonical projections. Next note that $\pi_1^{-1}(\pi_1(G_e)) = G_e P = VKP = VP$ is open implies that $\pi_1(G_e)$ is open and hence closed and so $\pi_1(G_e) = VP/P = (G/P)_e = W$ [2, Theorem 7.12]. That is, $W = VP/P$. Now consider the composition

$$G \xrightarrow{\pi_1} G/P = (VP/P)D \xrightarrow{\pi_2} VP/P \xrightarrow{\psi} V.$$

If $v \in V$, then $\psi(\pi_2(\pi_1(v))) = \psi(\pi_2(vP)) = \psi(vP) = v$. Thus $\pi = \psi \circ \pi_2 \circ \pi_1$ is a projection onto the normal subgroup V and $G = VL$ is a direct product with $L = \ker \pi$.

We now show that L is $\mathscr{B}$-invariant. Let $x \in L$ and let $O(x)$ be the closure of the $\mathscr{B}$-orbit of x which is compact and $\mathscr{B}$-invariant. Let G_x be the subgroup of G generated by $O(x)$. Then G_x is $\mathscr{B}$-invariant and so is a compactly generated $[FC]_{\mathscr{B}}^-$-group. By (2) $G_x \in [FD]_{\mathscr{B}}^-$. This means that the $\mathscr{B}$-commutator subgroup of G_x is compact so that its image in V under π is a compact, hence trivial, subgroup of V. It follows that $x^{-1}\beta(x) \in$ ker $\pi = L$ and $\beta(x) \in xL = L$, for each $\beta \in \mathscr{B}$. Since x was an arbitrary element of L, L is $\mathscr{B}$-invariant. Since $L_e = (G/V)_e = G_e/V = K$, L_e is compact [2, 7.13].

PROOF OF THE THEOREM.   Assume $G \in [FC]_{\mathscr{B}}^- \cap [SIN]_{\mathscr{B}}$. Then

$$G_e \in [FC]_{\mathscr{B}}^- \cap [SIN]_{\mathscr{B}}$$

so that the closure of $\mathscr{B}$ as a subgroup of $\mathfrak{A}(G_e)$ is compact [3, Theorem 0.1]. Since G_e is a connected $[SIN]$-group, it is maximally almost periodic and is a direct product $G_e = V_1 K_1$ of a vector group V_1 and a compact group K_1 [1, Théorème 16.4.6]. Since K_1 is a characteristic subgroup of G, there is an automorphism α of G_e such that $V = \alpha(V_1)$ is a $\mathscr{B}$-invariant subgroup of G and $G_e = VK_1$ [3, Theorem 1.1]. The lemma now applies and we have $G = VL$ with the desired properties. All that remains is to exhibit the required compact open subgroup K of L. The totally disconnected group L/L_e is in $[SIN]_{\mathscr{B}}$ so that any compact open subgroup K_2 in L/L_e contains a $\mathscr{B}$-invariant neighborhood of the identity. Thus $\bigcap \{\beta K_2 : \beta \in \mathscr{B}\}$ is a compact open $\mathscr{B}$-invariant subgroup of L/L_e. Let K be its inverse image in L.

Conversely, assume $G = VL$ as in the statement of the theorem. It suffices to show that $L \in [FC]_{\mathscr{B}}^-$. Let $\{x_\alpha K : \alpha \in A\}$ be a coset decomposition of the discrete group L/K. Let $x \in L$ so that $x = x_\alpha k$ for some α. The $\mathscr{B}$-orbit O of $x_\alpha K$ is finite. Thus, if π is the projection of L on L/K, we have $\pi(\beta(x_\alpha)) \in O$. Consequently, $\beta(x) = \beta(x_\alpha)\beta(k) \in \pi^{-1}(O)K$. That is, the $\mathscr{B}$-orbit of x is contained in a compact subset of L.

REMARKS.   The theorem stated above is a generalization of a structure theorem of Grosser and Moskowitz [3, Theorem 4.6]. In their case the group G was in $[FD]_{\mathscr{B}}^-$ and they were able to choose the compact subgroup K so that L/K was $\mathscr{B}$-fixed. That this is not generally possible for $G \in [FC]_{\mathscr{B}}^-$ is illustrated by considering a group G which is a discretely topologized weak direct sum of an infinite number of copies of a finite simple group [3, p. 39]. This group has finite conjugacy classes and the existence of such a (finite) subgroup K would imply that G had a finite commutator subgroup, which it does not.

Compactly generated locally compact abelian groups split as a direct product $R^n \times Z^m \times K$, with K compact [2, Theorem 9.8]. This theorem does not generalize to any reasonable class of nonabelian groups. However,

if we assume that G is a compactly generated group in $[SIN]_{\mathscr{B}}$, the theorem remains valid with "$L/K \simeq Z^m$ for some $m \geq 0$ and L/K is $\mathscr{B}$-fixed" replacing "$L/K \in [FC]_{\mathscr{B}}^-$". This can be obtained as a corollary of our theorem by utilizing the method of proof of [3, Proposition 4.5] as outlined below. Without loss of generality, we can now assume that K contains the $\mathscr{B}$-commutator subgroup of L so that L/K is a finitely generated abelian group and then enlarge K so that L/K is torsion-free and K is compact.

Our theorem has found applications in harmonic analysis. See [4].

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