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A SIMPLE ALTERNATIVE PROBLEM FOR FINDING PERIODIC SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL SYSTEMS[1]

J. W. BEBERNES

ABSTRACT. Existence of solutions for $x''=f(t, x, x')$, $x(0)=x(1)$, $x'(0)=x'(1)$ are proven by considering a simple alternative problem to which Leray-Schauder degree arguments can be directly applied.

1. **Introduction.** In this paper, we consider the existence of solutions to the periodic boundary value problem (PBVP)

$$x'' = f(t, x, x'), \tag{1}$$

$$x(0) = x(1), \qquad x'(0) = x'(1). \tag{2}$$

Knobloch [**4**], Mawhin [**5**], Schmitt [**6**], and Bebernes and Schmitt [**1**] have recently considered this problem using degree-theoretic arguments—either finite or infinite dimensional.

Using only the basic properties of Leray-Schauder degree and applying these degree arguments to a simple alternative problem associated with (1)-(2), we obtain in this paper a single basic result (Theorem 2.1) which contains and in some cases permits slight generalizations of most of the results of the above mentioned papers.

2. **The basic theorem.** Let $I=[0, 1]$, $\boldsymbol{R}^n$ be n-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and let $D\subset I\times \boldsymbol{R}^n\times \boldsymbol{R}^n$ be a bounded open set in the relative topology of $I\times \boldsymbol{R}^n\times \boldsymbol{R}^n$ containing $\{(t, 0, 0): t \in I\}$. Let $F: I\times \boldsymbol{R}^n\times \boldsymbol{R}^n\rightarrow \boldsymbol{R}^n$ be a continuous function and consider

$$x'' = F(t, x, x'). \tag{3}$$

For each $\lambda \in [0, 1]$, associate with (3) the equation

$$x'' = \lambda F(t, x, x') + (1 - \lambda)x. \tag{4}$$

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and assume:

(H) If $x(t)$ is a solution of (4)-(2), then $(t, x(t), x'(t)) \in D$ for all $t \in I$ or there exists $\tau \in I$ such that $(\tau, x(\tau), x'(\tau)) \notin \bar{D}$.

THEOREM. 2.1. *The periodic boundary value problem* (3)-(2) *has at least one solution such that* $(t, x(t), x'(t)) \in D$ *for all* $t \in I$.

PROOF. The periodic boundary value problem

$$x'' - x = 0, \qquad x(0) = x(1), \qquad x'(0) = x'(1) \tag{5}$$

has no nontrivial solutions. Let $H(t, x, x') = F(t, x, x') - x$, then $x(t)$ is a solution of (4)-(2) if and only if $x(t)$ is a solution of

$$x(t) = \lambda \int_0^1 G(t, s)H(s, x(s), x'(s))\, ds \tag{6}$$

where $G(t, s)$ is the unique Green's function for (5).

Let $B = \{x \in C'[0, 1]: x(0) = x(1),\ x'(0) = x'(1)\}$ with norm

$$|x| = \max_I \|x(t)\| + \max_I \|x'(t)\|$$

be the Banach space under consideration, and define

$$\Omega = \{y \in B: (t, y(t), y'(t)) \in D \text{ for all } t \in I\}.$$

Note that Ω is a bounded open subset of B.

Define the map $T: \bar{\Omega} \to B$ where $\bar{\Omega}$ is the closure of Ω by

$$(Ty)(t) = \int_0^1 G(t, s)H(s, y(s), y'(s))\, ds. \tag{7}$$

By standard arguments, $T(\bar{\Omega}) \subset B$, T is continuous, and $\mathrm{cl}(T(\bar{\Omega}))$ is compact in B.

If $0 \notin (I - \lambda T)(\partial\Omega)$ where $\partial\Omega$ is the boundary of Ω for all $\lambda \in [0, 1]$, then by the invariance under compact homotopy property of the Leray-Schauder degree [7, p. 92], the degree $\deg(I - \lambda T, \Omega, 0) = \text{constant}$ for all $\lambda \in [0, 1]$. That $0 \in (I - \lambda T)(\partial\Omega)$ is equivalent to the existence of a solution $x(t)$ of the PBVP (4)-(2) with $(t, x(t), x'(t)) \in \bar{D}$ for all $t \in I$ and $(t, x(t), x'(t)) \in \partial D$ for some $t \in I$; but by assumption (H) there exists no such solution $x(t)$ of (4)-(2) with $(t, x(t), x'(t)) \in \bar{D}$ for all $t \in I$ and $(t, x(t), x'(t)) \in \partial\bar{D}$ for some $t \in I$. Hence, $\deg(I - T, \Omega, 0) = \deg(I, \Omega, 0) = 1$. By the existence property of the Leray-Schauder degree [7, p. 88], there exists $x \in \Omega$ such that $(I - T)x = 0$. This means that there exists a solution $x(t)$ of the PBVP (3)-(2) with $(t, x(t), x'(t)) \in D$ for all $t \in I$.

3. **Applications of the basic theorem.** In this section, we illustrate how Theorem 2.1 can be used to prove existence results for PBVP (1)-(2).

The first result is known (e.g., [1], [4], or [5]), but it well illustrates the power of our basic theorem.

THEOREM 3.1. *If* $f(t, x, x')$ *is continuous on* $E_R=\{(t, x, x'): t \in I, \|x\|<R, \|x'\|<\infty\}$ *and satisfies*:

(8) $\|x'\|^2+\langle x, f(t, x, x')\rangle>0$ *for all* $(t, x, x') \in E_R$ *provided* $\|x\|=R$ *and* $\langle x, x'\rangle=0$;

(9) $\|f(t, x, x')\| \leqq \varphi(\|x'\|)$ *for all* $(t, x, x') \in E_R$ *where* φ *is a positive continuous function on* $[0, \infty)$ *with* $\int^\infty s/\varphi(s)\, ds=+\infty$;

(10) *there exists* $\alpha \geqq 0$, $K \geqq 0$ *such that*

$$\|f(t, x, x')\| \leqq 2\alpha(\|x'\|^2 + \langle x, f(t, x, x')\rangle) + K \quad \textit{for all } (t, x, x') \in E_R;$$

then there exists a solution $x(t)$ *of the PBVP* (1)-(2) *with* $(t, x(t), x'(t)) \in E_R$.

PROOF. Let $\delta_M(s)$ be a continuous function on $[0, \infty)$ with $\delta_M(s)=1$ on $[0, M]$ and $\delta_M(s)=0$ for $s \geqq 2M$ where $M=M(\alpha, K, R)$ is the Nagumo-Hartman bound (see Hartman [3, p. 429]).

Define

$$F(t, x, x') = \delta_M(\|x'\|) f(t, x, x') \qquad \text{on } E_R, \quad \text{and}$$
$$F(t, x, x') = (R/\|x\|) F(t, Rx/\|x\|, x') \quad \text{if } \|x\| \geqq R.$$

Then $F(t, x, x')$ is continuous and bounded on $I \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ and satisfies (8) provided $\|x\| \geqq R$ and $\langle x, x'\rangle=0$, (9), and (10) for all $(t, x, x') \in I \times \boldsymbol{R}^n \times \boldsymbol{R}^n$.

The proof will be completed by showing that there can be constructed an open bounded set $D \subset I \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ containing $\{(t, 0, 0): t \in I\}$ such that solutions of PBVP (4)-(2) satisfy hypothesis (H) relative to D.

For each $\lambda \in [0, 1]$, let $F_\lambda(t, x, x')=\lambda F(t, x, x')+(1-\lambda)F(t, x, x')$ where F is defined as above. Then for all $\lambda \in [0, 1]$ and all $(t, x, x') \in I \times \boldsymbol{R}^n \times \boldsymbol{R}^n$,

$$\|x'\|^2 + \langle x, F_\lambda(t, x, x')\rangle > 0 \quad \text{provided } \|x\| \geqq R \tag{11}$$

and $\langle x, x'\rangle=0$. Let $x(t)$ be any solution of (4)-(2). Define $u(t)=\|x(t)\|^2=\langle x(t), x(t)\rangle$. Because $u(t)$ satisfies the periodic boundary conditions (2), $u(t)$ can assume its maximum at $t_0 \in I$ only if $u(t_0)=0$, $u'(t_0) \leqq 0$. Claim $u(t)<R^2$ for all $t \in I$. Assume not; then there exists $t_0 \in I$ at which $u(t)$ assumes its maximum with $u(t_0) \geqq R^2$, $u'(t_0)=0$, and $u''(t_0) \leqq 0$. But (11) implies that $u''(t_0)>0$ which is a contradiction. Hence, $\|x(t)\|<R$ for all $t \in I$. For $(t, x, x') \in E_R$ and $\lambda \in [0, 1]$, $F_\lambda(t, x, x')$ is bounded which implies that $F_\lambda(t, x, x')$ satisfies a Nagumo-Hartman condition (conditions (9) and (10) with $\alpha=0$ and a K' in general different from K and $\varphi(s)=K'$). Hence, there exists an $M'>0$ such that if $x(t)$ is any solution of (4) on I with $\|x(t)\|<R$, then $\|x'(t)\|<M'$.

Define $D=\{(t, x, x'): t \in I, \|x\|<R, \|x'\|<M'\}$. From the observations made above it is immediate that solutions of (4)-(2) satisfy (H) relative to D. By Theorem 2.1, the PBVP (3)-(2) has a solution $x(t)$ with $\|x(t)\|<R$. Since $F(t, x, x')$ satisfies (9) and (10), $\|x'(t)\|<M$ on [0, 1] which implies that $x(t)$ is a solution of PBVP (1)-(2) on I with $(t, x(t), x'(t)) \in E_R$.

Equality can be permitted in (8) by an approximating argument like the one given in [3, p. 433].

The preceding theorem can be generalized by replacing $\|x\|^2$ by a function $V(t, x)$ which plays essentially the same role. In so doing, we obtain results similar to those obtained by Knobloch [4] and Mawhin [5].

Assume $f(t, x, x'): I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and let $\mathbf{R}^+$ denote the nonnegative reals.

Definition. Let $V \in C^2(I \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^+)$ be such that:

(a) there exists $R>0$ such that $\Phi \equiv \{x \in \mathbf{R}^n : V(t, x)<R, t \in I\}$ is bounded,

(b) $U(t, x, x') \equiv V_{tt}(t, x)+2\langle V_{tx}(t, x), x'\rangle+\langle V_{xx}(t, x)x', x'\rangle \geqq 0$,

(c) $V''_f(t, x)=U(t, x, x')+\langle V_x(t, x), f(t, x, x')\rangle>0$ provided $V(t, x)=R$ and $V_t(t, x)+\langle V_x(t, x), x'\rangle=0$,

(d) $\langle V_x(t, x), x\rangle>0$ for all (t, x) such that $V(t, x)=R$,

(e) $V(0, x)=V(1, x)$, $V_t(0, x)+\langle V_x(0, x), x'\rangle \geqq V_t(1, x)+\langle V_x(1, x), x'\rangle$.

Any such V is called a *bounding Lyapunov function* relative to (1).

Theorem 3.2. *If V is a bounding Lyapunov function for* (1), *then for every $\lambda \in [0, 1]$ every solution $x(t)$ of the PBVP*:

$$x''=f_\lambda(t, x, x') \tag{12}$$

where $f_\lambda=\lambda f+(1-\lambda)f$ *is such that $V(\tau, x(\tau))>R$ for some $\tau \in I$ or $V(t, x(t))<R$ for all $t \in I$.*

Proof. Let $x(t)$ be any solution of the PBVP (12)-(2) and let $m(t)=V(t, x(t))$, then $m'(t)=V_t(t, x(t))+\langle V_x(t, x(t)), x'(t)\rangle$ and

$$m''(t)=U(t, x(t), x'(t))+\langle V_x(t, x(t)), f_\lambda(t, x(t), x'(t))\rangle. \tag{13}$$

By (b), (c), and (d), $m''(t)>0$ if $V(t, x(t))=R$ and $V_t(t, x(t))+\langle V_x(t, x(t)), x'(t)\rangle=0$. If there exists $\tau \in I$ such that $m(\tau)>R$, we are through.

Assume $m(t) \leqq R$ for all $t \in [0, 1]$. If there exists $t_0 \in I$ such that $m(t_0)=R$, then $m'(t_0)=0$ and $m''(t_0) \leqq 0$ since $m(0)=m(1)$ and $m'(0) \geqq m'(1)$ by (e). But this is impossible by the observation made above that $m''(t_0)>0$. Hence, $m(t)<R$ on I and the conclusion of the theorem follows.

Our next theorem is similar to Theorem 6.1 [5].

THEOREM 3.3. *If V is a positive definite bounding Lyapunov function relative to* (1) *and if there exists $S>0$ such that for any $\lambda \in [0, 1]$ any solution $x(t)$ of PBVP* (12)-(2) *with $V(t, x(t))<R$ on I satisfies $\|x'(t)\|<S$ for $t \in I$, then PBVP* (1)-(2) *has at least one solution $x(t)$ with $V(t, x(t))<R$.*

PROOF. Let $D=\{(t, x, x'): t \in I,\ V(t, x)<R,\ \|x'\|<S\}$. By Theorem 3.2, solutions of (12)-(2) satisfy (H) relative to D. Hence, by Theorem 1.2, the conclusion follows.

There are several ways of ensuring the a priori bound condition on the derivative of solutions of (12)-(2) and hence we have the following corollaries.

COROLLARY 3.4. *If V is a bounding positive definite Lyapunov function for* (1) *and if $f(t, x, x')$ satisfies* (9) *and* (10) *for all $t \in I$, $x \in \Phi$, $\|x'\|<\infty$, then PBVP* (1)-(2) *has a solution $x(t) \in \Phi$ for all $t \in I$.*

COROLLARY 3.5. *If V is a bounding positive definite Lyapunov function for* (1), *if $f(t, x, x')$ satisfies* (9) *for all $t \in I$, $x \in \Phi$, $\|x'\|<\infty$, and if there exists $\beta \geqq 0$, $L \geqq 0$ such that*

$$\|f(t, x, x')\| \leqq \beta(U(t, x, x') + \langle V_x(t, x), f(t, x, x')\rangle) + L \tag{14}$$

for all $t \in I$, $x \in \Phi$, and $\|x'\| \leqq \infty$, then PBVP (1)-(2) *has a solution $x(t) \in \Phi$ for all $t \in I$.*

COROLLARY 3.6. *If V is a bounding positive definite Lyapunov function for* (1), *if $f(t, x, x')$ satisfies* (9) *for all $t \in I$, $x \in \Phi$, $\|x'\|<\infty$, and if there exists a function $\rho(t) \in C^2(I)$ such that*

$$\|f(t, x, x')\| \leqq \rho''(t) \quad \textit{for all } t \in I,\ x \in \Phi,\ \|x'\| < \infty, \tag{15}$$

then PBVP (1)-(2) *has a solution $x(t) \in \Phi$ for all $t \in I$.*

4. **Further consequences.** In this section, we present two further applications of Theorem 2.1. The first theorem presented shows that the bounding set Φ need not be given in terms of a bounding Lyapunov function. Assume $f(t, x, x'): I \times R^n \times R^n \to R^n$ is continuous.

THEOREM 4.1. *Let G be a bounded convex open set in R^n containing* 0 *and assume there is a function $N: \partial G \to R^n$ satisfying:*

$$\langle N(x), x\rangle > 0 \quad \textit{for all } x \in \partial G, \tag{16}$$

$$\bar{G} \subseteq \{y: \langle N(x), y - x\rangle \leqq 0 \quad \textit{for each } x \in \partial G\}, \tag{17}$$

$$\langle N(x), f(t, x, x')\rangle > 0 \quad \textit{for all } t \in I,\ x \in \partial G,\ x' \textit{ with } \langle N(x), x'\rangle = 0, \tag{18}$$

then for every $\lambda \in [0, 1]$ every solution $x(t)$ of (12)-(2) *is such that $x(\tau) \notin G$ for some $\tau \in I$ or $x(t) \in G$ for all $t \in I$.*

REMARK. Conditions (16) and (17) say that $N(x)$ is an outer normal for G. Gustafson and Schmitt [2] have used a similar outer normal condition to study existence of periodic solutions for delay differential equations.

PROOF. Let $x(t)$ be any solution of (12)-(2). If $x(\tau) \notin G$ for some $\tau \in I$, we are through so assume $x(t) \in \bar{G}$ for all $t \in I$.

If $x(t_0) \in \partial G$ for some $t_0 \in I$, we may assume $t_0 \in [0, 1)$. By (16) and (18), $\langle N(x(t_0)), f_\lambda(t_0, x(t_0), x'(t_0))\rangle > 0$ and hence there is an $h>0$ such that $\langle N(x(t_0)), x''(t)\rangle > 0$ for all $t \in [t_0, t_0+h)$. Since $x(t) \in \bar{G}$, $\langle N(x(t_0)), x'(t_0)\rangle = 0$. Looking at the Taylor expansion for $x(t)$, we have immediately that

$$\begin{aligned}&\langle N(x(t_0)), x(t) - x(t_0)\rangle \\ &\qquad = (t - t_0)\langle N(x(t_0)), x'(t_0)\rangle + \tfrac{1}{2}(t - t_0)^2\langle N(x(t_0)), y(\bar{\xi})\rangle\end{aligned}$$

where $y(\bar{\xi}) = (x_1''(\xi), \cdots, x_n''(\xi_n))$ and $t_0 < \xi_i < t < t_0 + h$ for all $i = 1, \cdots, n$. From this, $\langle N(x(t_0)), x(t) - x(t_0)\rangle > 0$ meaning that $x(t) \notin \bar{G}$, which is a contradiction.

Our existence theorem then follows.

THEOREM 4.2. *If G is a bounded convex open set in $\boldsymbol{R}^n$ containing* 0, *if there is a function $N: \partial G \to \boldsymbol{R}^n$ satisfying* (16), (17), *and* (18), *and if there exists $S>0$ such that for any $\lambda \in [0, 1]$ any solution $x(t)$ of PBVP* (12)-(2) *with $x(t) \in G$ for all $t \in I$ satisfies $\|x'(t)\| < S$ for $t \in I$, then PBVP* (1)-(2) *has at least one solution with $x(t) \in G$ for all $t \in I$.*

PROOF. Let $D = \{(t, x, x') : t \in I,\ x \in G,\ \|x'\| < S\}$. By Theorem 4.1 solutions of (11)-(2) satisfy (H) relative to D. Result then follows immediately from Theorem 2.1.

REMARK. One can state corollaries of the above theorem analogous to Corollaries 3.4, 3.5, and 3.6.

In $\boldsymbol{R}^n$, let $x \leqq y$ if and only if $x_i \leqq y_i$, $1 \leqq i \leqq n$, and $x < y$ if and only if $x_i < y_i$, $1 \leqq i \leqq n$.

Let $f(t, x, x')$ be continuous on $\{(t, x, x') : t \in I, \alpha(t) \leqq x \leqq \beta(t), x' \in \boldsymbol{R}^n\}$ where $\alpha, \beta : I \to \boldsymbol{R}^n$, $\alpha(t) < 0 < \beta(t)$ are twice continuously differentiable with

$$\alpha(0) = \alpha(1), \quad \beta(0) = \beta(1), \quad \alpha'(0) \geqq \alpha'(1), \quad \beta'(0) \leqq \beta'(1). \tag{19}$$

Assume also that α, β are strict lower, upper solutions of (1), i.e.,

$$\begin{aligned}&\alpha''(t) > f_i(t, x_1, \cdots, x_{i-1}, \alpha_i(t), x_{i+1}, \cdots, x_n, x_1', \cdots, \\ &\qquad\qquad\qquad\qquad x_{i-1}', \alpha_i'(t), x_{i+1}', \cdots, x_n'), \\ &\beta_i''(t) < f_i(t, x_1, \cdots, x_{i-1}, \beta_i(t), x_{i+1}, \cdots, x_n, x_1', \cdots, \\ &\qquad\qquad\qquad\qquad x_{i-1}', \beta_i'(t), x_{i+1}', \cdots, x_n'),\end{aligned} \tag{20}$$

and

$$\alpha_i''(t) > \alpha_i(t), \qquad \beta_i''(t) < \beta_i(t) \tag{21}$$

for $\alpha_j(t) \leqq x_j \leqq \beta_j(t), j \neq i, i=1, \cdots, n$.

We now can state our final result.

THEOREM 4.3. *If f is continuous on $\{(t, x, x'): t \in I,\ \alpha(t) \leqq x \leqq \beta(t),\ x' \in \mathbf{R}^n\}$ where α, β are strict periodic lower, upper solutions of* (1) *satisfying* (19), (20), *and* (21), *and if there exists $S>0$ such that for any $\lambda \in [0, 1]$ any solution $x(t)$ of* (12)-(2) *with $\alpha(t) \leqq x(t) \leqq \beta(t)$ on I satisfies $\|x'(t)\| < S$ then PBVP* (1)-(2) *has a solution $x(t)$ with $\alpha(t) < x(t) < \beta(t)$.*

The proof is similar to those previously given and is for this reason omitted. By a proper modification of $f(t, x, x')$, condition (21) can be dropped and equality can be permitted in (20). With that observation, we have a generalization of Theorem 4.1 in **[1]**.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302