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UNIFORM CONVERGENCE FOR A HYPERSPACE

R. J. GAZIK

ABSTRACT. In this note a uniform convergence in the collection $C(E)$ of nonempty, compact subsets of a separated uniform convergence space E is defined. This convergence is compared with the hyperspace convergence on $C(E)$ and it is shown that the two convergences agree on Richardson's class Γ. In the case of a regular T_1 topological space (E, t) this means that there is a uniform convergence structure on E, which induces t, such that uniform convergence in $C(E)$ is convergences in the Vietoris topology on $C(E)$.

1. **Definition of uniform convergence.** Let $C(E)$ be the collection of nonempty, compact subsets of a Hausdorff topological space (E, t). Then $C(E)$ may be equipped with the Vietoris topology. If, in addition, (E, t) is completely regular and $\mathscr{U}$ is a uniform structure which induces t, then $C(E)$ also carries the uniform topology. A classical result is that the Vietoris and uniform topologies agree on $C(E)$ (see [6]).

Now let $C(E)$ be the collection of nonempty, compact subsets of a Hausdorff convergence space (E, t) (see [3]). There is a reasonable way to generalize the Vietoris and uniform topologies to this setting. In fact, in [4] a convergence $h(t)$ for $C(E)$ was defined. It was shown that $h(t)$ is the Vietoris topology for topological t. Moreover $h(t)$ is Hausdorff (regular) (compact) whenever t is Hausdorff (regular) (compact and regular). But no additional properties of t (such as complete regularity) are needed to make a reasonable definition of uniform convergence in $C(E)$ and this is done below.

Let $(E, \mathscr{J})$ be a separated uniform convergence space (see [2]). In [1] Cochran defined a U^* base for $\mathscr{J}$ to be a base β for $\mathscr{J}$ such that each member of β is coarser than the diagonal filter, each member of β is its own inverse, the composition of any two members of β exists and is finer than a third member, and the infimum of two members of β is again in β. Each uniform convergence space has a U^* base; put $\beta=(J \in \mathscr{J}: J \leqq [\Delta], J=J^{-1})$. It should also be pointed out that each Hausdorff topological space (E, t), indeed each Hausdorff convergence space (E, t), has a uniform convergence structure $\mathscr{J}$, constructively defined, such that

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$t(\mathscr{J})$—the convergence induced by $\mathscr{J}$—is the same as t (Cochran [**1**] and Keller [**5**]).

REMARK. With respect to the definition below, notice that the convergence defined depends on $\mathscr{J}$ but not on the choice of a U^* base for $\mathscr{J}$.

DEFINITION. Let $C(E)$ be the collection of nonempty, $t(\mathscr{J})$ compact subsets of a separated uniform convergence space $(E, \mathscr{J})$ and let β be a U^* base for $\mathscr{J}$. For $J \in \beta$, $V \in J$, define $T(J, V)$ to be the set of all ordered pairs (A, B), $A, B \in C(E)$, such that $A \subset V(B)$, $B \subset V(A)$, and define $T(J)$ to be the filter generated by the $T(J, V)$, $V \in J$. The structure $\mathscr{J}$ of uniform convergence in $C(E)$ is the uniform convergence structure generated by the $T(J)$, $J \in \beta$.

THEOREM 1. *Let $(E, \mathscr{J})$ be a separated uniform convergence space. Then*

(a) *$\mathscr{J}$ indeed is a uniform convergence structure for $C(E)$;*

(b) *if $\mathscr{J}$ is a uniform structure then $t(\mathscr{J})$ is the uniform topology;*

(c) *a filter Φ on $C(E)$ converges relative to $t(\mathscr{J})$ to $A \in C(E)$ if and only if $\Phi \times \dot{A} \geqq T(J)$ for some $J \in \beta$.*

PROOF. (a) Since β is a U^* base, each member of β contains the diagonal so $T(J, V) \neq \varnothing$ and $T(J)$ is a filter. If D is the diagonal filter in $C(E) \times C(E)$ then $D \geqq T([\Delta])$ so $D \in \mathscr{J}$. Each member of β is its own inverse, hence $(T(J))^{-1} \geqq T(J)$. Since each member of $\mathscr{J}$ is finer than a finite infimum of $T(J)$'s, it follows that the inverse of a member of $\mathscr{J}$ is again in $\mathscr{J}$. Finally, notice that if $J, J_1, J_2, \cdots, J_n \in \beta$ then

(1) $T(J) \circ (\bigwedge T(J_i)) \geqq \bigwedge T(J \circ J_i \wedge J_i \circ J)$. Since each $J \circ J_i \wedge J_i \circ J$ is finer than a member of β we may apply (1) repeatedly to show that the composition of two members of $\mathscr{J}$ is again in $\mathscr{J}$ when it exists.

(b) If $\mathscr{J}$ is a uniform structure, $\mathscr{J}$ itself is a U^* base of one element and it is obvious that $t(\mathscr{J})$ is the uniform topology.

(c) This follows from the definition, the fact that β is a U^* base and the inequality $\bigwedge T(J_i) \geqq T(\bigwedge J_i)$.

2. **Comparison of $t(\mathscr{J})$ and $h(t(\mathscr{J}))$.** If $C(E)$ is the collection of nonempty $t(\mathscr{J})$ compact subsets of a separated uniform convergence space $(E, \mathscr{J})$, there are two convergences defined on $C(E)$, namely the convergence $t(\mathscr{J})$ defined in §1 and the hyperspace convergence $h(t(\mathscr{J}))$ of [**4**] with respect to the convergence $t(\mathscr{J})$ induced by $\mathscr{J}$. Theorem 2 below compares these in general.

Notice that a filter α in a uniform convergence space $(E, \mathscr{J})$ $t(\mathscr{J})$ accumulates at $x \in E$ if and only if there exists $J \in \mathscr{J}$, such that $F \in \alpha$ and $V \in J$ implies $F \cap V(x) \neq \varnothing$. We abbreviate this by saying "α $t(\mathscr{J})$ accumulates at x with respect to J".

THEOREM 2. *If* $(E, \mathscr{J})$ *is a separated uniform convergence space,* $t(\mathscr{J}) \geqq h(t(\mathscr{J}))$.

PROOF. Let $\Phi \to A$ relative to $t(\mathscr{J})$ so that $\Phi \times \dot{A} \geqq T(J)$ for some $J \in \beta$, β a U^* base for $\mathscr{J}$. In order to show that Φ $h(t(\mathscr{J}))$ converges to A it must be proved that (1) and (2) of Definition 2.1 of [4] hold.

PROOF OF (1). Suppose that (x, g) is a selection of a cofinal segment (D, f) of Φ. Let $V \in J$ and $d \in D$.

(a) There exists $r(V) \in \Phi$ such that $B \subset V(A)$ for each $B \in r(V)$.

(b) There exists $p(d, V) \geqq d$ such that $f(p(d, V)) \subset r(V)$. The statements (a) and (b) come, respectively, from the convergence of Φ and the fact that (D, f) is a cofinal segment of Φ. Since (x, g) is a selection of (D, f), (a) and (b) imply

(c) $(x(p(d, V)), a(p(d, V))) \in V$ for some $a(p(d, V)) \in A$. Direct $D \times J$ with the product order; that is $(d, V) \geqq (e, U)$ if and only if $d \geqq e$ and $V \subset U$. Then $(a(p(d, V)) : (d, V) \in D \times J)$ is a net in A whose section filter accumulates at some $a \in A$ with respect to some $K \in \beta$. (A is compact.)

Now let $V \in J$, $W \in K$, $d \in D$. From the remarks above there is an $(e, U) \geqq (d, V)$ such that $(a(p(e, U)), a) \in W$. By (c)

$$(x(p(e, U)), a(p(e, U))) \in U \subset V.$$

These last two statements mean that $(x(p(e, U)), a) \in V \circ W$ with $p(e, U) \geqq d$. This in turn means that the section filter $S(x)$ $t(\mathscr{J})$ accumulates at a with respect to $J \circ K \in \mathscr{J}$. Hence $S(x)$ $t(\mathscr{J})$ accumulates at a point of A.

PROOF OF (2). Let $a \in A$. If $(V, r) \in J \times \Phi$ use the convergence of Φ to see that there is an $f(V, r) \subset r$, $f(V, r) \in \Phi$, such that $A \subset V(B)$ whenever $B \in f(V, r)$. With $J \times \Phi$ ordered with the product order, $(J \times \Phi, f)$ is a cofinal segment of Φ. Also, clearly, $J(a) \to a$. It is asserted that $(J \times \Phi, f)$ and $J(a)$ satisfy (2) of Definition 2.1 of [4]. For, if $V(a) \in J(a)$ and $r \in \Phi$ there is a $U \in J$ such that $U^{-1} \subset V$. ($J = J^{-1}$ for $J \in \beta$.) Whenever $B \in f(W, s)$, $(W, s) \geqq (U, r)$, the relation $A \subset W(B) \subset U(B)$ is a consequence of the properties of f so $(a, b) \in U$ for some $b \in B$. Then $(b, a) \in U^{-1} \subset V$, hence $B \cap V(a) \neq \varnothing$. This completes the proof.

Without restrictions on $\mathscr{J}$ we cannot hope to obtain equality in Theorem 2, even if very strong conditions are placed on $t(\mathscr{J})$. The following illustrates the point.

EXAMPLE. Let (E, t) be a Hausdorff topological space and let S be a finite subset of E. Define $T(S) = \bigwedge (N(x) \times N(x) : x \in S) \wedge D$ where $N(x)$ is the neighborhood filter at x and D is the diagonal filter. It is easy to see that the collection of $T(S)$, S finite, generates a uniform convergence structure $\mathscr{J}$ on E and, in fact, $t(\mathscr{J}) = t$.

In particular, let (E, t) be the closed unit disk with the usual topology. By Theorem 2.2 of [4], $h(t(\mathscr{J}))$ is the Vietoris topology on $C(E)$. But notice that $t(\mathscr{J})$ is not compact, even though $(E, \mathscr{J})$ is. Hence uniform convergence in $C(E)$ is not the same as the hyperspace convergence $h(t(\mathscr{J}))$.

With respect to this example and Theorem 3.7 of [4] one might reasonably ask the following question. If $(E, \mathscr{J})$ is compact and regular, does there exist a uniform convergence structure $\mathscr{K}$ for E, with $t(\mathscr{K})=t(\mathscr{J})$, such that $t(\mathscr{K})$ is compact? The author does not know the answer.

Richardson [7] defined a class Γ of uniform convergence structures for E as follows: $\mathscr{J} \in \Gamma$ if and only if there exists $J \in \mathscr{J}$ such that whenever $x \in E$ and $\alpha \to x$, then $\alpha \geqq J(x)$. *We will call such a J a fixed member of $\mathscr{J}$ for convenience.* A fixed member of $\mathscr{J}$ may be taken to be symmetric and coarser than the diagonal filter.

LEMMA 1. *Let $\mathscr{J} \in \Gamma$ with J a fixed member of $\mathscr{J}$. If A is a compact subset of $(E, \mathscr{J})$ and $V \in J$, then $A \subset V(F)$ for some finite subset F of A.*

PROOF. If not, $A \cap (V(F))' \neq \varnothing$ for each finite subset F of A. The filter generated by the $A \cap (V(F))'$ accumulates at some point $a \in A$ with respect to the same J. Then $A \cap (V(a))' \cap V(a) \neq \varnothing$—a contradiction.

THEOREM 3. *If $(E, \mathscr{J})$ is separated and $\mathscr{J} \in \Gamma$ then $t(\mathscr{J})=h(t(\mathscr{J}))$.*

PROOF. Let β be a U^* base for $\mathscr{J}$ and suppose $\Phi \to A$ relative to $h(t(\mathscr{J}))$. In order to show that $\Phi \to A$ relative to $t(\mathscr{J})$ it suffices to prove the following.

(a) For some $K \in \mathscr{J}$, whenever $U \in K$ there exists $r \in \Phi$ such that $B \subset U(A)$ for all $B \in r$.

(b) For some $L \in \mathscr{J}$, whenever $U \in L$ there exists $r \in \Phi$ such that $A \subset U(B)$ for all $B \in r$.

PROOF OF (a). Suppose (a) is false for $K=J$, J a fixed member of $\mathscr{J}$. There is some $V \in J$ such that, whenever $r \in \Phi$, $g(r) \not\subset V(A)$ for some $g(r) \in r$. Hence there exists $x(r) \in g(r) - V(A)$. If Φ is ordered by $r \geqq s$ if $r \subset s$, and f is the identity map on Φ, then (x, g) is a selection of the cofinal segment (Φ, f) of Φ. Since $\Phi \to A$ relative to $h(t(\mathscr{J}))$, $S(x)$—the section filter of x—accumulates at some $a \in A$ with respect to J (for $\mathscr{J} \in \Gamma$). Thus, if r is arbitrary in Φ there is an $s \subset r$ for which $x(s) \in V(a)$. But this contradicts $x(s) \notin V(A)$.

PROOF OF (b). We assert that (b) is true for $L=J \circ J$, J a fixed member of $\mathscr{J}$. By Lemma 1:

(1) If $V \in J$, $A \subset V(F)$ for some finite subset $F \subset A$. The following is also true.

(2) If $a \in A$, $V \in J$, there exists $r(a, V) \in \Phi$ such that whenever $B \in r(a, V)$, then $a \in V(B)$.

For suppose (2) is false. For $r \in \Phi$ there exists $g(r) \in r$ with $a \notin V(g(r))$. The filter α of Definition 2.1 of [**4**] is finer than $J(a)$ so that, for some r, $g(r) \cap U(a) \neq \varnothing$, $U^{-1} \subset V$, $U \in J$. From this, there is some $y \in g(r)$ such that $a \in U^{-1}(y) \subset V(y) \subset V(g(r))$. This is a contradiction and (2) holds.

Finally, let $V \circ V \in J \circ J$. Use (1) and (2) and put $r = \bigcap\ (r(a, V) : a \in F)$. Let $B \in r$ and $z \in A$. Then $z \in V(a)$ for some $a \in F$ and $a \in V(B)$ so $z \in (V \circ V)(B)$ follows and we conclude that $A \subset (V \circ V)(B)$. This completes the proof.

Theorem 4 is an immediate consequence of the result above, Theorem 2.2 of [**4**], and the major result of [7]. In this regard notice that in [7] Richardson actually constructs $\mathscr{J}$ from the neighborhood filters, so that $\mathscr{J}$ is constructively defined.

THEOREM 4. *Let $C(E)$ be the collection of nonempty, compact subsets of a regular T_1 topological space (E, t). There exists a uniform convergence structure $\mathscr{J}$ for E, with $t(\mathscr{J}) = t$, such that $t(\mathscr{J})$ is the Vietoris topology on $C(E)$.*

Necessary and sufficient conditions on $\mathscr{J}$ which insure $t(\mathscr{J}) = h(t(\mathscr{J}))$ would be very interesting. The author knows only the partial result in Theorem 3.

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DEPARTMENT OF MATHEMATICS, ARKANSAS STATE UNIVERSITY, STATE UNIVERSITY, ARKANSAS 72467

Current address: Department of Mathematics, Washington State University, Pullman, Washington 99163