A COUNTABLY DISTRIBUTIVE COMPLETE BOOLEAN ALGEBRA NOT UNCOUNTABLY REPRESENTABLE

JOHN GREGORY[1]

ABSTRACT. It is proved from the Continuum Hypothesis that there exists an ω-distributive complete Boolean algebra which is not ω_1-representable.

Karp [1, Corollary to main theorem] shows that there is an ω-distributive non-$|2^\omega|$-representable $|2^\omega|$-complete Boolean algebra. Karp later pointed out that, for every cardinal $K \geq |2^\omega|$, there is an ω-distributive non-$|2^\omega|$-representable K-complete Boolean algebra. The following strengthens the completeness requirement.

THEOREM 1. *The Continuum Hypothesis implies the existence of an ω-distributive non-ω_1-representable complete Boolean algebra.*

In [3, Definition 1.2], the property P_K was defined for cardinals K. We will prove the following theorem.

THEOREM 2. *If $|2^\omega| < |2^{\omega_1}|$, then there exists an ω-distributive complete Boolean algebra that does not have the property P_{ω_1}.*

Theorem 1 is a consequence of Theorem 2, since the Continuum Hypothesis implies both $|2^\omega| < |2^{\omega_1}|$ and (by [3, Theorem 4.2]) the equivalence of ω_1-representability with property P_{ω_1}.

Assuming the negation of Souslin's Hypothesis, Smith's [3, Theorem 3.4] implies our theorems. However, R. Jensen has shown that Souslin's Hypothesis is consistent with the Continuum Hypothesis; so Smith's result does not imply ours.

1. **Preliminaries.** Let ω be the first infinite cardinal; ω_1 is the first uncountable cardinal; x^y is the set of all functions from y into x; $|x|$ is the cardinal of x (cardinals are initial ordinals); $Dm(x)$ is the domain of function x; $x {\restriction} y$ is the restriction of a function x to the domain $Dm(x) \cap y$. We assume the axiom of choice.

The Boolean algebraic definitions are essentially those of [2]. A Boolean algebra C is ordered by: $a \leq b$ iff the infimum $a \wedge b = a$. C is K-complete if the infimum (or meet) $\wedge S$ exists for each set S of no more than K many elements of C. Then the supremum (or join) $\vee S$ exists for all such S. C is complete if the infimum of every set of elements exists. A K-complete Boolean algebra C is K-distributive iff

$$\bigwedge_{\alpha < K} \bigvee_{\beta < K} A(\alpha, \beta) \leq \bigvee_{H \in K^K} \bigwedge_{\alpha < K} A(\alpha, H(\alpha))$$

for all functions A into C. A Boolean algebra is K-representable iff it is isomorphic to the quotient of a K-field of sets by a K-ideal. Smith defined the following distributive-like property P_K which implies K-representability.

DEFINITION. A Boolean algebra C has the property P_K iff: if A is a function from $K \times K$ into C and if $\bigwedge_{\alpha < K} \bigvee_{\beta < K} A(\alpha, \beta)$ exists and is not zero, then there is a function H from K into K such that, for all $\gamma < K$, either $\bigwedge_{\alpha < \gamma} A(\alpha, H(\alpha))$ does not exist or else is not zero.

A subset S of C is dense iff S does not contain the zero of C and, for all nonzero elements b of C, there exists $x \in S$ such that $x \leq b$.

The following is well known and is related to "forcing".

A set S ordered by $\leq$ satisfies

(1) for all $x \nleq y$, there exists $z \leq x$ such that,
     for all w, not both $w \leq y$ and $w \leq z$

if and only if there exists an (algebraically unique) complete Boolean algebra C such that S is a dense subset of C and the ordering of C extends that of S. (See [2, Example 12(B)] and [2, §35].)

If S is dense in C and x, y, z range over S, then: $x \leq \wedge X$ iff, for all $b \in X$, $x \leq b$; $x \leq$ complement of b iff, for all z, not both $z \leq x$ and $z \leq b$; $x \leq \vee X$ iff, for all $y \leq x$, there is $z \leq y$ and $b \in X$ such that $z \leq b$.

2. **Proof of Theorem 2.** Assume $\Omega = |2^\omega| < |2^{\omega_1}|$. Suppose there did not exist an ω-distributive complete Boolean algebra not having property P_{ω_1}. It will suffice to reach a contradiction. Let X be the set of all nonempty countable sequences f into 2^ω (i.e., of all functions f whose nonempty domain is a countable ordinal and whose range is a subset of 2^ω).

Define $\mathcal{F}$ to be the set of all F such that: F is a function from a subset of X into X; if $F(f)$ defined, then $\mathrm{Dm}(F(f)) = \mathrm{Dm}(f) + 1$; if $F(f)$ defined, then so is $F(f \restriction \alpha) = F(f) \restriction (\alpha + 1)$ whenever $0 < \alpha \leq \mathrm{Dm}(f)$.

There exists a Ω-sequence E that enumerates the set of all ordered pairs (f, t) such that $f \in X$ and countable $t \subseteq X$, each such pair occurring Ω times in the sequence E.

By induction on $\alpha \leq \Omega$, F_α will now be constructed such that $F_\alpha \in \mathscr{F}$, $|F_\alpha| \leq |\omega \cup \alpha|$, and F_β extends F_α whenever $\alpha < \beta \leq \Omega$.

For limit ordinal $\lambda \leq \Omega$, let F_λ be the union of all F_α, $\alpha < \lambda$. (In particular, F_0 is empty.)

Given $F = F_\alpha$, $\alpha < \Omega$, $G = F_{\alpha+1}$ is defined by cases as follows. In each case below, note that $G \in \mathscr{F}$, G extends F, and G has only countably more elements than F. (Cases 1, 2, 3 are for Lemmas 1(1), 1(2), 2 respectively.)

Case 1. $E(\alpha)$ is some $(f, 0)$ such that $\mathrm{Dm}(f) = 2$. Since $(2^\omega)^1$ has cardinality greater than F_α, there is $h \in (2^\omega)^1$ such that $F(h)$ is not defined. Extend F to G by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

Case 2. $E(\alpha)$ is some $(f, \{g\})$ such that f has a domain $\gamma + 1$, g has a domain $\delta < \gamma$, and $F(g) = f{\restriction}(\delta+1)$. Then there is $h \in (2^\omega)^\gamma$ such that h extends g and $F(h{\restriction}(\delta+1))$ is not defined. Then $F(h{\restriction}\beta)$ is defined iff $0 < \beta \leq \delta$; for such β, $h{\restriction}\beta$ is $g{\restriction}\beta$. Extend F to G by: $G(h{\restriction}\beta) = f{\restriction}(\beta+1)$ whenever $\delta < \beta \leq \gamma$; $G(k) = F(k)$ whenever defined.

Case 3. $E(\alpha)$ is some (f, t) such that: f has a domain $\gamma + 1$; for every $k \in t$, $F(k) \subseteq f$ is defined; there are Ω many $h \in (2^\omega)^\gamma$ such that h is a union of elements of t. Then there exists $h \in (2^\omega)^\gamma$ such that h is a union of elements of t but $F(h)$ is not defined. Extend F to G by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

Case 4.   The first three cases do not hold. Let G be F.

This finishes the construction of F_α, $\alpha \leq \Omega$. From now on, let F be F_Ω.

Given $B \in (2^\omega)^{\omega_1}$, define T to be $\{f \mid F(f) \subseteq B\}$.

T will turn out to be a dense subset of a complete ω-distributive Boolean algebra. But first some facts about T will be proved.

LEMMA 1.   (1) *T is nonempty.* (2) *If g is in both T and $(2^\omega)^\delta$ and if $\delta < \gamma < \omega_1$, then g has at least two distinct extensions in $T \cap (2^\omega)^\gamma$.* (3) *If $f \in T$ and nonzero $\beta \leq \mathrm{Dm}(f)$, then $f{\restriction}\beta \in T$.*

PROOF.   (1) For some α, $E(\alpha)$ is $(B{\restriction}2, 0)$. By Case 1 of the construction of $F_{\alpha+1}$, there is h such that $F_{\alpha+1}(h) = B{\restriction}2$. Then $F(h) = B{\restriction}2$ and this h is in T.

(2) Then for some $\beta < \Omega$, $F_\beta(g) = B{\restriction}(\delta+1)$ is defined. For Ω many $\alpha > \beta$, $E(\alpha)$ is $(B{\restriction}(\gamma+1), \{g\})$. For each such α, by Case 2 of the construction of $F_{\alpha+1}$, there is assigned $h \in (2^\omega)^\gamma$ such that h extends g, $F_{\alpha+1}(h) = B{\restriction}(\gamma+1)$, and $F_\alpha(h)$ is not defined. Then h is in $T \cap (2^\omega)^\gamma$ and h extends g. Since $F_\alpha(h)$ is not defined, h was not assigned to a smaller α; thus, distinct such α's give distinct h's.

(3) $F(f) \subseteq B$. Thus $F(f)$ is defined. Since $F \in \mathscr{F}$, $F(f{\restriction}\beta) \subseteq F(f) \subseteq B$. Thus, $f{\restriction}\beta \in T$.

LEMMA 2.   *If countable $t \subseteq T$ and there are Ω many $h \in (2^\omega)^\gamma$ such that h is a union of elements of t, then there exists $h \in T \cap (2^\omega)^\gamma$ such that h is a union of elements of t.*

PROOF.   For each $k \in t$, $F(k) \subseteq B$ and some $F_\beta(k)$, $\beta < \Omega$, is defined. Since Ω is not cofinal with ω, there exists $\beta < \Omega$ such that, for all $k \in t$, $F_\beta(k)$ is defined. For some $\alpha > \beta$, $E(\alpha)$ is $(B {\restriction} (\gamma+1), t)$. Then Case 3 of the definition of $F_{\alpha+1}$ gives $h \in (2^\omega)^\gamma$ such that h is a union of elements of t and $F_{\alpha+1}(h) = B {\restriction} (\gamma+1)$. Then $F(h) \subseteq B$ and $h \in T$.

Order T by: $f \leqq g$ iff f extends g. Then T satisfies (1) of §1. So there is a (algebraically unique) complete Boolean algebra C such that T is a dense subset of C and the ordering of C extends that of T.

LEMMA 3.   *C is an ω-distributive Boolean algebra.*

PROOF.   Suppose that A is a function on $\omega \times K$ into C. By the definition of ω-distributivity and by the fact that T is dense in C, it suffices to show for every $f \in T$ that: if $f \leqq \bigwedge_m \bigvee_\alpha A(m, \alpha)$, then there is $h \in T$ such that $h \leqq f$ and, for all $m < \omega$, $h \leqq A(m, \alpha)$ for some $H(m) = \alpha < K$.

Define $P(r) \in T$ for $r \in 2^m$ by the following induction on $m < \omega$. Let $P(0) = f$. Suppose $P(r) \leqq f$ defined for all $r \in 2^m$. Let $\beta(m)$ be the least upper bound of $\{ \mathrm{Dm}(P(r)) | r \in 2^m \}$. Consider any $r \in 2^m$. Let u and v be the two extensions in 2^{m+1} of r. Since $P(r) \leqq f \leqq \bigvee_\alpha A(m, \alpha)$, there exists $g \in T$ such that $g \leqq P(r)$ and $g \leqq A(m, \alpha)$ for some α. By Lemma 1(2), there exist distinct extensions $P(u)$ and $P(v)$ of g such that they both have the same domain greater than $\beta(m)$.

Put $t = \{ P(r) | r \in 2^m$ for some $m < \omega \}$. Let $\gamma = \bigcup_m \beta(m)$. By Lemma 2, there exists $h \in T \cap (2^\omega)^\gamma$ such that h is a union of some elements of t. Then $h \leqq P(0) = f$. Consider any $m < \omega$. There is $r \in 2^{m+1}$ such that $P(r) \subseteq h$. There is $\alpha < K$ such that $h \leqq P(r) \leqq A(m, \alpha)$.

LEMMA 4.   *T has an uncountable chain.*

PROOF.   Define A on $\omega_1 \times 2$ into C by

$$A(\omega\beta + m, i) = \bigvee \{ f \in T | f(\beta)(m) = i \}, \quad \beta < \omega_1, m < \omega, i < 2.$$

By Lemma 1(2), for every $g \in T$ there is $f \leqq g$ such that $\beta \in \mathrm{Dm}(f)$ and hence $f \leqq A(\omega\beta + m, f(\beta)(m))$. Thus since T is dense, it follows that each $\bigvee_i A(\omega\beta + m, i)$ is the identity of C. Since C is a complete ω-distributive Boolean algebra, it must have property P_{ω_1} (by our earlier supposition, which is used here only). By P_{ω_1}, there exists a function H on ω_1 into 2 such that, for all $\gamma < \omega_1$, the infimum $\bigwedge_{\beta < \gamma} A(\beta, H(\beta))$ is nonzero. Let $J = \{ f \in X | f(\beta)(m) = H(\omega\beta + m)$ for all $\beta < \mathrm{Dm}(f)$ and $m < \omega \}$. Then J is an uncountable chain of functions. It now suffices to prove $J \subseteq T$. Consider any $f \in J$. Let $\alpha = \mathrm{Dm}(f)$ and $\gamma = \omega\alpha$. Since T is dense, there exists $g \in T$,

$g \leqq \bigwedge_{\beta < \gamma} A(\beta, H(\beta))$. Then $g(\beta)(m) = H(\omega\beta + m)$ for all $\beta < \alpha$, $m < \omega$; so $g|\alpha$ is f. By Lemma 1(3), $f \in T$.

For each $B \in (2^\omega)^{\omega_1}$, we now choose $T(n, B)$, $J(n, B)$, by induction on $n < \omega$, such that $T(n, B) \subseteq X$, $J(n, B) \in (2^\omega)^{\omega_1}$, and $J(n, B)|\alpha \in T(n, B)$ for all nonzero $\alpha < \omega_1$.

$T(0, B) = X$. $J(0, B) = B$.

Suppose $T(n, B)$ and $J(n, B)$ are defined. Define $T(n+1, B) = \{f | F(f) \subset J(n, B)\}$ (i.e., define $T(n+1, B)$ from $J(n, B)$ as T was earlier defined from B). By Lemma 4, $T(n+1, B)$ has an uncountable branch. The union of such a branch is some element $J(n+1, B)$ of $(2^\omega)^{\omega_1}$.

LEMMA 5. *If B and B' are such that, for all $n < \omega$, $J(n, B)|1$ equals $J(n, B')|1$, then B equals B'.*

PROOF. Let B and B' be such functions. It suffices to prove, by induction on nonzero $\beta \leqq \omega_1$, that $R(\beta):J(n, B)|B = J(n, B')|\beta$ for all $n < \omega$. For $\beta = 1$, $R(\beta)$ is given. For nonzero limit β, $R(\beta)$ follows easily from $R(\alpha)$ for all $\alpha < \beta$. Suppose $R(\beta)$ holds for nonzero β, to show $R(\beta+1)$. Consider any $n < \omega$. Let f be $J(n+1, B)|\beta$. By $R(\beta)$, f equals $J(n+1, B')|\beta$. Then $F(f) \subset J(n, B)$ since $f \in T(n+1, B)$; $F(f) \subset J(n, B')$ since $f \in T(n+1, B')$. Since $F(f)$ has domain $\beta+1$, it follows that $J(n, B)|(\beta+1) = F(f) = J(n, B')|(\beta+1)$.

By Lemma 5, distinct $B \in (2^\omega)^{\omega_1}$ are assigned distinct sequences $\langle J(n, B)|1 | n < \omega \rangle$. Thus, $|(2^\omega)^{\omega_1}| \leqq |((2^\omega)^1)^\omega|$. This contradicts the assumption that $|2^{\omega_1}| > |2^\omega|$. Theorem 2 is proved.

It follows from the proof of Theorem 2 that: if $|2^\omega| < |2^{\omega_1}|$, then, for some B, the corresponding T defined above is a dense subset of an ω-distributive complete Boolean algebra not having property P_{ω_1}.

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4517 JONWALL COURT, COLUMBIA, SOUTH CAROLINA 29206

Current address: Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226