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ON THE JOIN OF SUBNORMAL SUBGROUPS

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ABSTRACT. Let $\mathfrak{G}$ be the class of finitely generated groups. If the join of finitely many subnormal $\mathfrak{X}=sn\mathfrak{X}$ subgroups is always an $\mathfrak{X}$-group and $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}\subseteq\mathfrak{G}$, then the join of finitely many subnormal $\mathfrak{X}\mathfrak{Y}$-subgroups is an $\mathfrak{X}\mathfrak{Y}$-group. If the subnormal coalition class $\mathfrak{X}$ and the class $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}$ are such that whenever $A\in\mathfrak{X}\mathfrak{Y}$, A has a maximum subnormal $\mathfrak{X}$-subgroup, then $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$ is a subnormal coalition class ($\mathfrak{Y}\wedge\mathfrak{G}$ is the class of finitely generated $\mathfrak{Y}$-groups).

1. **Introduction and notation.** In this section we state our results. The notation used is discussed in 1.3 and 1.4.

1.1. DEFINITION. The class $\mathfrak{X}$ is a subnormal coalition class if, whenever H and K are subnormal $\mathfrak{X}$-subgroups of G, their join $\langle H, K\rangle$ is a subnormal $\mathfrak{X}$-subgroup of G.

We establish a condition which implies that the class $\mathfrak{X}_1\mathfrak{X}_2$ is a subnormal coalition class, given that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are subnormal coalition classes.

THEOREM A. *If the subnormal coalition class $\mathfrak{X}$ and the class $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}$ are such that whenever $A\in\mathfrak{X}\mathfrak{Y}$, A has a maximum subnormal $\mathfrak{X}$-subgroup, then $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$ is a subnormal coalition class.*

We may take, for example, for the class $\mathfrak{Y}$ of Theorem A the class $\hat{\mathfrak{M}}_s$.

It is a consequence of 2.8 that whenever $\mathfrak{X}$ is a subnormal coalition class and $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}\subseteq\mathfrak{G}$, an $\mathfrak{X}\mathfrak{Y}$-group has a maximum subnormal $\mathfrak{X}$-subgroup. Hence, as a corollary to Theorem A we have

THEOREM B. *If $\mathfrak{X}$ is a subnormal coalition class and $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}\subseteq\mathfrak{G}$ is a class of groups, then $\mathfrak{X}\mathfrak{Y}$ is a subnormal coalition class.*

In Theorem B we may take for the class $\mathfrak{Y}$ the classes $\mathfrak{F}$, $\hat{\mathfrak{M}}$, and $\mathfrak{G}^{sn}$.

1.2. DEFINITION. If $\mathfrak{X}$ is a class of groups, then $G\in s_0\mathfrak{X}$ if and only if G is the join of finitely many subnormal $\mathfrak{X}$-subgroups.

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It is clear that every subnormal coalition class is s_0-closed. It is shown in [4] that the class of solvable groups is s_0-closed.

We also investigate conditions which imply that $\mathfrak{X}_1\mathfrak{X}_2$ is s_0-closed, given that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are s_0-closed classes.

THEOREM C. *If* $\mathfrak{X}=\{sn, s_0\}\mathfrak{X}$ *and* $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}\subseteq\mathfrak{G}$ *then* $\mathfrak{X}\mathfrak{Y}=s_0\mathfrak{X}\mathfrak{Y}$.

We leave as an open question whether the condition "$\mathfrak{X}=sn\mathfrak{X}$" may be deleted from the hypothesis of Theorem C. A result in this direction is

THEOREM D. *If* $\mathfrak{X}=s_0\mathfrak{X}$, *then* $\mathfrak{X}\mathfrak{F}=s_0\mathfrak{X}\mathfrak{F}$.

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1.3. The identity element and the group of order one are denoted by 1. If H is a subgroup of G, we write $H\subseteq G$ and denote by $|G:H|$ the index of H in G. If $|G:H|=n$ and $G=\bigcup_{i=1}^{n} Ha(i)$, we say that the set $T=\{a(1), a(2),\cdots, a(n)\}$ is a right transversal for H in G. If H is a subnormal (normal) subgroup of G, we write $H\lhd\lhd G$ ($H\lhd G$) and denote by $s(G, H)$ the subnormal index for H in G. If G is generated by the subsets T_α, $\alpha\in A$, we write $G=\langle T_\alpha|\alpha\in A\rangle$. H^g denotes the conjugate of H by $g\in G$. If T is a subset of G, $H^T=\langle H^t|t\in T\rangle$. If H and K are subgroups of G, HK is the set $\{hk|h\in H,\ k\in K\}$ and $[H, K]=\langle [h, k]|h\in H, k\in K\rangle$, where $[h, k]=h^{-1}k^{-1}hk$. Define $[H, {}_nK]$ inductively by $[H, {}_0K]=H$ and $[H, {}_{i+1}K]=[[H, {}_iK], K]$.

1.4. A class of groups is a collection of groups $\mathfrak{X}$ such that $1\in\mathfrak{X}$ and whenever $G\in\mathfrak{X}$ and G_1 is isomorphic to G, then $G_1\in\mathfrak{X}$. We let

$\mathfrak{F}$ = the class of finite groups,
$\mathfrak{G}$ = the class of finitely generated groups,
$\mathfrak{G}^{sn}$ = the class of groups, all of whose subnormal subgroups are finitely generated,
$\hat{\mathfrak{M}}(\hat{\mathfrak{M}}_s)$ = the class of groups satisfying the maximal condition for subgroups (subnormal subgroups).

If $\mathfrak{X}$ is a class of groups, we let

$sn\mathfrak{X}$ = class of subnormal subgroups of $\mathfrak{X}$-groups,
$q\mathfrak{X}$ = class of quotients of $\mathfrak{X}$-groups,
$n_0\mathfrak{X}$ = class of products of finitely many normal $\mathfrak{X}$-subgroups.

If $\varphi\in\{sn, q, n_0, s_0\}$, we say that the class $\mathfrak{X}$ is φ-closed if $\varphi\mathfrak{X}=\mathfrak{X}$. If $Y\subseteq\{sn, q, n_0, s_0\}$, $\mathfrak{X}=Y\mathfrak{X}$ if $\mathfrak{X}$ is φ-closed for all $\varphi\in Y$. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of groups, $\mathfrak{X}\mathfrak{Y}$ denotes the class of $\mathfrak{X}$-by-$\mathfrak{Y}$ groups and $\mathfrak{X}\wedge\mathfrak{Y}$ denotes the intersection of $\mathfrak{X}$ and $\mathfrak{Y}$.

2. Some preliminaries.

2.1. LEMMA [1, LEMMA 3.21]. *Let H and K be subgroups of a group, let $N \triangleleft K$ and suppose K/N can be generated by n elements. Then, for any $t>0$, $H^K=L^N[H, {}_tK]$ where L is generated by at most $1+n+n^2+\cdots+n^{t-1}$ conjugates of H by elements of K.*

2.2. LEMMA [2, LEMMA 2.4]. *Let $\mathfrak{X}=\{sn, n_0\}\mathfrak{X}$. If $H \triangleleft\triangleleft G$, $K \triangleleft\triangleleft G$, $J=\langle H, K\rangle$ and $J=HK$, then $J \triangleleft\triangleleft G$; also $J \in \mathfrak{X}$ if $H \in \mathfrak{X}$ and $K \in \mathfrak{X}$.*

We will need the main results of [3].

2.3. DEFINITION [3, p. 423]. The class $\mathfrak{X}$ is locally coalescent if whenever H and K are subnormal $\mathfrak{X}$-subgroups of G, then every finitely generated subgroup F of $J=\langle H, K\rangle$ is contained in some subnormal $\mathfrak{X}$-subgroup X of G such that $F \subseteq X \subseteq J$.

2.4. THEOREM [3, THEOREMS A AND B]. *If $\mathfrak{X}$ is a class of groups such that $\mathfrak{X}=\{sn, n_0\}\mathfrak{X}$, then $\mathfrak{X}$ is locally coalescent. If $\mathfrak{X}$ is a locally coalescent class, then $\mathfrak{X} \wedge \mathfrak{G}$ is a subnormal coalition class.*

2.5. DEFINITION. If $\mathfrak{X}$ is a class of groups, then

$$\theta_{\mathfrak{X}}(G) = \langle H \mid H \triangleleft\triangleleft G \text{ and } H \in \mathfrak{X}\rangle.$$

2.6. The following are immediate consequences of Definition 2.5:

(i) $\theta_{\mathfrak{X}}(G)$ is a characteristic subgroup of G.

(ii) If K is a finitely generated subgroup of $\theta_{\mathfrak{X}}(G)$, there exist finitely many subnormal $\mathfrak{X}$-subgroups $H_1, H_2, \cdots, H_n$ of G such that $K \subseteq \langle H_1, H_2, \cdots, H_n\rangle$.

(iii) If $\mathfrak{X}=sn\mathfrak{X}$ is a subnormal coalition class and $K \triangleleft\triangleleft G$, then $\theta_{\mathfrak{X}}(K)=\theta_{\mathfrak{X}}(G) \cap K$.

2.7. LEMMA [3, p. 424]. *Let $\mathfrak{X}$ be a locally coalescent class. If $\theta_{\mathfrak{X}}(G)=G$, then every finitely generated subgroup of G is contained in some subnormal $\mathfrak{X}$-subgroup of G.*

2.8. LEMMA. *Let $\mathfrak{X}=s_0\mathfrak{X}$ and let $\mathfrak{Y}=\{sn, q\}\mathfrak{Y} \subseteq \mathfrak{G}$. If $G \in \mathfrak{X}\mathfrak{Y}$, then $\theta_{\mathfrak{X}}(G) \in \mathfrak{X}$ and $G/\theta_{\mathfrak{X}}(G) \in \mathfrak{Y}$.*

PROOF. Let $G \in \mathfrak{X}\mathfrak{Y}$ and let $N \triangleleft G$ such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. Since $N \subseteq \theta_{\mathfrak{X}}(G)$, $\theta_{\mathfrak{X}}(G)/N \in \mathfrak{Y} \subseteq \mathfrak{G}$. It follows from 2.6(ii) that $\theta_{\mathfrak{X}}(G) \in s_0\mathfrak{X}=\mathfrak{X}$. Since $q\mathfrak{Y}=\mathfrak{Y}$, we have $G/\theta_{\mathfrak{X}}(G) \in \mathfrak{Y}$. □

2.9. LEMMA. *If $\mathfrak{X}=\{sn, n_0\}\mathfrak{X}$ and $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}$ are classes such that whenever $A \in \mathfrak{X}\mathfrak{Y}$, $\theta_{\mathfrak{X}}(A) \in \mathfrak{X}$, then the class $\mathfrak{X}\mathfrak{Y}$ is locally coalescent.*

PROOF. By 2.4, it suffices to show that $\mathfrak{X}\mathfrak{Y}=\{sn, n_0\}\mathfrak{X}\mathfrak{Y}$.

Let H and K be normal $\mathfrak{X}\mathfrak{Y}$-subgroups of $J=\langle H, K\rangle$ and let $F_H=\theta_{\mathfrak{X}}(H)$ and $F_K=\theta_{\mathfrak{X}}(K)$. By hypothesis, $F_H, F_K \in \mathfrak{X}$ and H/F_H, $K/F_K \in q\mathfrak{Y}=\mathfrak{Y}$. It follows that F_H and F_K are normal $\mathfrak{X}$-subgroups of J and $F_HF_K \in \mathfrak{X}=n_0\mathfrak{X}$. Hence, $J/F_HF_K \in \{q, n_0\}\mathfrak{Y}=\mathfrak{Y}$ and $\mathfrak{X}\mathfrak{Y}=n_0\mathfrak{X}\mathfrak{Y}$.

If $H\lhd G\in\mathfrak{X}\mathfrak{Y}$, then $K=\theta_{\mathfrak{X}}(G)\in\mathfrak{X}$ and $G/K\in\mathfrak{Y}$. Hence, $HK/K\lhd G/K$ and $HK/K\in\mathfrak{Y}=sn\mathfrak{Y}$. Also, $H\cap K\in\mathfrak{X}=sn\mathfrak{X}$. It follows that $H\in\mathfrak{X}\mathfrak{Y}$ and $\mathfrak{X}\mathfrak{Y}=sn\mathfrak{X}\mathfrak{Y}$. □

3. Subnormal coalition classes.

3.1. LEMMA. *Let $\mathfrak{X}$ be a subnormal coalition class and let $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}$ be a class of groups such that whenever $A\in\mathfrak{X}\mathfrak{Y}$, $\theta_{\mathfrak{X}}(A)\in\mathfrak{X}$. Let H and K be subnormal $\mathfrak{X}\mathfrak{Y}$-subgroups of G such that $\langle H, K\rangle=HK$. If $N\lhd H$ such that $N\in\mathfrak{X}$, $H/N\in\mathfrak{Y}$, and $\langle N, K\rangle$ is a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup of G, then $\langle H, K\rangle$ is a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup of G.*

PROOF. It follows from 2.2 that HK is subnormal in G.

Let $M=\langle N, K\rangle$ and let

$$M = M_m \lhd M_{m-1} \lhd \cdots \lhd M_1 \lhd M_0 = HK$$

be the standard series for M in HK. Suppose $M_{i+1}\in\mathfrak{X}\mathfrak{Y}$. Then $\theta_{\mathfrak{X}}(M_{i+1})=\bar{M}\in\mathfrak{X}$ and $\bar{M}\lhd M_i$. Now,

$$M_i = M_i \cap HK = (M_i \cap H)M_{i+1}$$

and

$$M_i/\bar{M} = ((M_i \cap H)\bar{M}/\bar{M})(M_{i+1}/\bar{M}).$$

Since $(M_i\cap H)\bar{M}/\bar{M}$ and $M_{i+1}/\bar{M}$ are subnormal $\mathfrak{Y}=\{sn, n_0\}\mathfrak{Y}$-subgroups of $M_i/\bar{M}$, it follows from 2.2 that $M_i/\bar{M}\in\mathfrak{Y}$. Hence, $M_i\in\mathfrak{X}\mathfrak{Y}$ and $HK\in\mathfrak{X}\mathfrak{Y}$. □

3.2. LEMMA. *Let $\mathfrak{X}$ be a subnormal coalition class and let $\mathfrak{Y}=\{sn, q, n_0\}\mathfrak{Y}$ be a class of groups such that whenever $A\in\mathfrak{X}\mathfrak{Y}$, $\theta_{\mathfrak{X}}(A)\in\mathfrak{X}$. If H is a subnormal $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$-subgroup and K is a subnormal $\mathfrak{X}$-subgroup of G, then $\langle H, K\rangle$ is a subnormal $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$-subgroup of G.*

PROOF. Let $N=\theta_{\mathfrak{X}}(H)$, $J=\langle H, K\rangle$, and $\bar{J}=\langle K^H, N\rangle$. By hypothesis, $N\in\mathfrak{X}$ and $H/N\in q(\mathfrak{Y}\wedge\mathfrak{G})=\mathfrak{Y}\wedge\mathfrak{G}$. If $t=s(G, H)$, it follows from 2.1 that

$$\bar{J} = \langle L^N[K, {}_tH], N\rangle = \langle L, N\rangle\langle[K, {}_tH], N\rangle,$$

where L is the join of a finite number of conjugates of K. Since $\mathfrak{X}$ is a subnormal coalition class, $M=\langle L, N\rangle$ is a subnormal $\mathfrak{X}$-subgroup of G. Since $N\subseteq\langle[K, {}_tH], N\rangle\lhd H$ and $\mathfrak{Y}=q\mathfrak{Y}$, $\langle[K, {}_tH], N\rangle$ is a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup of G. Consequently, by 2.2, $\bar{J}\lhd\lhd G$ and $J=\bar{J}H\lhd\lhd G$. An application of 3.1 shows that $\bar{J} \in \mathfrak{X}\mathfrak{Y}$. A second application of 3.1 shows that $J=\bar{J}H \in \mathfrak{X}\mathfrak{Y}$. Since $\theta_{\mathfrak{X}}(J) \in \mathfrak{X}$ and $J/\theta_{\mathfrak{X}}(J) \in \mathfrak{G}$, $J \in \mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$. $\square$

PROOF OF THEOREM A. Let H and K be subnormal $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$-subgroups of G and let $J=\langle H, K\rangle$. Let $N\lhd H$ such that $N \in \mathfrak{X}$ and $H/N \in \mathfrak{Y}\wedge\mathfrak{G}$.

If $t=s(G, H)$, it follows from 2.1 that

$$\langle K^H, N\rangle = \langle L^N[K, {}_tH], N\rangle = \langle L, N\rangle\langle [K, {}_tH], N\rangle,$$

where L is the join of a finite number of conjugates of K. By induction on $s(G, K)$ we conclude that L is a subnormal $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$-subgroup of G. An application of 3.2 shows that $\langle L, N\rangle$ is a subnormal $\mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$-subgroup of G. Also, $\langle [K, {}_tH], N\rangle \in \mathfrak{X}\mathfrak{Y}$. It follows from 3.1 that $\langle K^H, N\rangle \in \mathfrak{X}\mathfrak{Y}$ and from 2.2 that $\langle K^H, N\rangle \lhd\lhd G$. Since $J=\langle K^H, N\rangle H$, it follows from 3.1 that $J \in \mathfrak{X}\mathfrak{Y}$ and from 2.2 that $J\lhd\lhd G$. Since $\theta_{\mathfrak{X}}(J) \in \mathfrak{X}$ and $J/\theta_{\mathfrak{X}}(J) \in \mathfrak{G}$, $J \in \mathfrak{X}(\mathfrak{Y}\wedge\mathfrak{G})$. $\square$

4. s_0-closed classes.

4.1. LEMMA. *Let $\mathfrak{X}=s_0\mathfrak{X}$ and $\mathfrak{Y}=\{sn, q\}\mathfrak{Y} \subseteq \mathfrak{G}$ be two classes of groups. If $H=\langle H_1, H_2, \cdots, H_n\rangle$, where H_i is a subnormal $\mathfrak{X}$-subgroup of G, and K is a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup of G, then $\langle H, K\rangle$ is an $\mathfrak{X}\mathfrak{Y}$-subgroup of G.*

PROOF. Since $K \in \mathfrak{X}\mathfrak{Y}$, there exists $N\lhd K$ such that $N \in \mathfrak{X}$ and $K/N \in \mathfrak{Y}$. If $t=s(G, K)$, an application of 2.1 shows that $\langle H^K, N\rangle=\langle L, [H, {}_tK], N\rangle$, where L is the join of a finite number of conjugates of H. Since $[H, {}_tK]\lhd K$, $[H, {}_tK]N/N \in \mathfrak{Y}=sn\mathfrak{Y} \subseteq \mathfrak{G}$ and there exist finitely many elements $x_1, x_2, \cdots, x_l \in [H, {}_tK]$ such that

$$\langle [H, {}_tK], N\rangle = \langle x_1, x_2, \cdots, x_l, N\rangle.$$

Since $[H, {}_tK] \subseteq H^K$, there exist finitely many elements $k_1, k_2, \cdots, k_m \in K$ such that

$$\langle x_1, x_2, \cdots, x_l\rangle \subseteq \langle H^{k_1}, H^{k_2}, \cdots, H^{k_m}\rangle.$$

Consequently,

$$\langle H^K, N\rangle = \langle L, H^{k_1}, H^{k_2}, \cdots, H^{k_m}, N\rangle \in \mathfrak{X} = s_0\mathfrak{X}.$$

But then $\langle H, K\rangle/\langle H^K, N\rangle \in q\mathfrak{Y}=\mathfrak{Y}$ and $\langle H, K\rangle \in \mathfrak{X}\mathfrak{Y}$. $\square$

PROOF OF THEOREM C. Let $G=\langle H_1, H_2, \cdots, H_n\rangle$, where H_i is a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup of G, $1\leq i\leq n$. Let $F_i=\theta_{\mathfrak{X}}(H_i)$. By 2.8, $F_i \in \mathfrak{X}$ and $H_i/F_i \in \mathfrak{Y} \subseteq \mathfrak{G}$. Let T_i be a finite subset of H_i such that $H_i=\langle F_i, T_i\rangle$ and let $T=\bigcup_{i=1}^n T_i$.

Since $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy the hypothesis of 2.9, the class $\mathfrak{X}\mathfrak{Y}$ is locally coalescent. There exists by 2.7 a subnormal $\mathfrak{X}\mathfrak{Y}$-subgroup K of G such that $\langle T\rangle \subseteq K$. An application of 4.1 shows that $\langle F_1, F_2, \cdots, F_n, K\rangle = G \in \mathfrak{X}\mathfrak{Y}$. $\square$

4.2. LEMMA. *Suppose $H \lhd G$, $|G:H| = n < \infty$, and $K \subseteq H$. If $A = \{1 = a(1), a(2), \cdots, a(n)\}$ is a right transversal for H in G such that $G = \langle K, A\rangle$, then there exists a finite subset L of H such that $H = \langle K^a, L \mid a^{-1} \in A\rangle$.*

PROOF. Let $a, b \in A$ and $k \in K$. Since $K \subseteq H \lhd G$, we see that $akb^{-1} \in H$ if and only if $a = b$. For all $a(i), a(j) \in A$, $1 \leqq i, j \leqq n$, we define $a(i,j) \in A$ uniquely by the equation $a(i)a(j)a(i,j)^{-1} \in H$.

Let $\bar{H}$ be defined by

$$\bar{H} = \langle K^a, a(i)a(j)a(i,j)^{-1} \mid a^{-1} \in A, 1 \leqq i, j \leqq n\rangle.$$

Since $H \lhd G$, $\bar{H} \subseteq H$. If $g \in H$, then $g = g_1^{\varepsilon_1} g_2^{\varepsilon_2} \cdots g_m^{\varepsilon_m}$ for some elements $g_i \in K \cup A$ and $\varepsilon_i = \pm 1$, $1 \leqq i \leqq m$. Set $a(i_0) = 1$. There exists a unique element $a(i_1) \in A$ such that $a(i_0)g_1^{\varepsilon_1}a(i_1)^{-1} \in H$. It is easily verified that $(a(i_0)g_1^{\varepsilon_1}a(i_1)^{-1})^{\varepsilon_1}$ is a displayed generator of $\bar{H}$. Suppose that for all j, $1 \leqq j < l \leqq m$, we have chosen $a(i_j)$ such that $(a(i_{j-1})g_j^{\varepsilon_j}a(i_j)^{-1})^{\varepsilon_j}$ is a generator of $\bar{H}$. We then choose $a(i_l) \in A$ as the unique element satisfying the equation $a(i_{l-1})g_l^{\varepsilon_l}a(i_l)^{-1} \in H$. Again, $(a(i_{l-1})g_l^{\varepsilon_l}a(i_l)^{-1})^{\varepsilon_l}$ is a generator of $\bar{H}$. But then

$$g = a(i_0)g_1^{\varepsilon_1}a(i_1)^{-1}a(i_1)g_2^{\varepsilon_2}a(i_2)^{-1} \cdots a(i_{m-1})g_m^{\varepsilon_m}a(i_m)^{-1}a(i_m),$$

where $a(i_{l-1})g_l^{\varepsilon_l}a(i_l)^{-1} \in \bar{H}$, $1 \leqq l \leqq m$. Since $\bar{H} \subseteq H$, $a(i_m) = 1$. Hence, $g \in \bar{H}$ and $H = \bar{H}$. The lemma follows if we set $L = \{a(i)a(j)a(i,j)^{-1} \mid 1 \leqq i, j \leqq n\}$. $\square$

PROOF OF THEOREM D. Let $G = \langle H_1, H_2, \cdots, H_n\rangle$ where H_i is a subnormal $\mathfrak{X}\mathfrak{F}$-subgroup of G, $1 \leqq i \leqq n$. Let $F_i = \theta_{\mathfrak{X}}(H_i)$. By 2.8, $F_i \in \mathfrak{X}$ and $H_i/F_i \in \mathfrak{F}$. Since $F_i \subseteq \theta_{\mathfrak{X}}(G)$, it follows that $H_i\theta_{\mathfrak{X}}(G)/\theta_{\mathfrak{X}}(G)$ is a finite subnormal subgroup of $G/\theta_{\mathfrak{X}}(G)$. It is a consequence of 2.4 that $\mathfrak{F}$ is a subnormal coalition class. Consequently, $G/\theta_{\mathfrak{X}}(G) \in \mathfrak{F}$.

Let A and A_i, $1 \leqq i \leqq n$, be right transversals for $\theta_{\mathfrak{X}}(G)$ in G and F_i in H_i respectively such that $1 \in A$. Since $G = \langle H_1, H_2, \cdots, H_n\rangle$,

$$G = \langle F_1, F_2, \cdots, F_n, A_1, A_2, \cdots, A_n\rangle.$$

There exists a finite subset U of $\theta_{\mathfrak{X}}(G)$ such that $G = \langle F_1, F_2, \cdots, F_n, U, A\rangle$. By 2.6(ii), there exist a finite number of subnormal $\mathfrak{X}$-subgroups $L_1, L_2, \cdots, L_l$ of G such that $\langle U\rangle \subseteq \langle L_1, L_2, \cdots, L_l\rangle$. If we let $K = \langle F_i, L_j \mid 1 \leqq i \leqq n, 1 \leqq j \leqq l\rangle$, an application of 4.2 shows the existence of a finite subset V of $\theta_{\mathfrak{X}}(G)$ such that

$$\theta_{\mathfrak{X}}(G) = \langle K^a, V \mid a^{-1} \in A\rangle.$$

By 2.6, there exist a finite number of subnormal $\mathfrak{X}$-subgroups $M_1, M_2, \cdots, M_m$ of G such that $\langle V\rangle \subseteq \langle M_1, M_2, \cdots, M_m\rangle$. Hence, $\theta_{\mathfrak{X}}(G)$ is the join of a finite number of subnormal $\mathfrak{X}$-subgroups and $\theta_{\mathfrak{X}}(G) \in \mathfrak{X} = s_0\mathfrak{X}$. $\square$

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