PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

PROBLEM 26 OF L. FUCHS

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ABSTRACT. This solves the following problem: Which Abelian groups are the inverse limits of Abelian groups, each of which is a finite direct sum of quasi-cyclic and bounded Abelian groups? (Here quasi-cyclic means isomorphic to some $Z(p^\infty)$.) A necessary and sufficient condition for an Abelian group to be such is that it takes the form $A_r \oplus \Pi_p \operatorname{Hom}_Z(A_p, Z(p^\infty))$ where A_r is complete and reduced, the A_p are torsion-free and the direct product is taken over the set of prime numbers.

We are going to solve the following problem of L. Fuchs [1]: Which Abelian groups are the inverse limits of Abelian groups each of which is a finite direct sum of quasi-cyclic and bounded Abelian groups?

We shall adopt the following notations for an Abelian group A: A_d is its maximal divisible subgroup; $A_r = A/A_d$; $A[n] = \{x \in A \mid nx = 0\}$; $T_p(A)$ is the p-primary component of the torsion subgroup of A. We let Z denote the group of integers, Q the rational numbers, $\hat{Z}_p$ the p-adic integers, and $Z(p^\infty) = T_p(Q/Z)$. An Abelian group is *quasi-cyclic* if it is isomorphic to $Z(p^\infty)$ for some prime number p. To say that A is a finite direct sum of quasi-cyclic and bounded Abelian groups is equivalent to the conditions: (a) A_d is a finite direct sum of quasi-cyclic groups; (b) A_r is bounded.

Let R be a ring. An R-module shall mean a left R-module. A topology on an R-module A shall be one in which the additive group of A becomes a (Hausdorff) topological group. It is *linear* if there is an open base at 0 consisting of R-submodules. A linear topology on A is *linearly compact* if it satisfies the condition: Given a family $\{K_\omega\}_{\omega \in \Omega}$ of residue classes of A modulo closed R-submodules, if every finite subfamily has a nonempty intersection then $\bigcap_{\omega \in \Omega} K_\omega \neq \varnothing$.

Suppose that A is an R-module and A' is an R-submodule with some topology. For $x \in A$, we call a subset of $x + A'$ a *linear subset* if it has the form $y + B$, where $y \in x + A'$ and B is a closed R-submodule of A. Evidently we have

LEMMA 1. *Let A, B be R-modules, A', B' be respectively their submodules with some topologies, and $\varphi: A \to B$ be an R-homomorphism which induces a continuous R-homomorphism $A' \to B'$.*

Received by the editors November 16, 1970 and, in revised form, November 22, 1971.
AMS (MOS) subject classifications (1970). Primary 20K25.

(i) *If* $\{K_\omega\}_{\omega\in\Omega}$ *is a family of linear subsets of* $x+A'$ $(x\in A)$, *then* $\bigcap_{\omega\in\Omega} K_\omega$ *is either a linear subset or* $\varnothing$.

Furthermore, if A' *is either linearly compact or compact and every finite subfamily of* $\{K_\omega\}_{\omega\in\Omega}$ *has nonempty intersection then* $\bigcap_{\omega\in\Omega} K_\omega\neq\varnothing$.

(ii) *For* $y\in\varphi(x)+B'$, $\varphi^{-1}(y)\cap(x+A')$ *is a linear subset of* $x+A'$.

(iii) *If* A' *is either linearly compact or compact and* K *is a linear subset of* $x+A'$ $(x\in A)$, *then* $\varphi(K)$ *is a linear subset of* $\varphi(x)+B'$.

The next lemma is due to C. U. Jensen [2]. An alternative proof is included here because it is more elementary and needs fewer assumptions than that of [2]. The idea of this proof is derived from [3, Proposition 13-2-1, p. 66]. (We were not aware of the result of [2] until Professor Joseph Rotman kindly informed us. We are also indebted to the referee for some improvements.)

LEMMA 2 (C. U. JENSEN). *Let*

$$0 \longrightarrow \{A'_\alpha, \pi'_{\alpha\beta}\} \xrightarrow{\{\sigma_\alpha\}} \{A_\alpha, \pi_{\alpha\beta}\} \xrightarrow{\{\tau_\alpha\}} \{A''_\alpha, \pi''_{\alpha\beta}\} \longrightarrow 0$$

be an exact sequence of inverse systems of R-modules where A'_α *are linearly compact (in some linear topologies) and* $\pi'_{\alpha\beta}$ *are continuous, then* $\tau=\operatorname{proj\,lim}\tau_\alpha$ *is onto.*

The same conclusion also holds if A'_α are compact (in some topologies) instead of linearly compact.

PROOF. Given $x=\{x_\alpha\}\in\operatorname{proj\,lim} A''_\alpha$ we have an inverse system of sets $\{E_\alpha, f_{\alpha\beta}\}$, where $E_\alpha=\tau_\alpha^{-1}(x_\alpha)$ and $f_{\alpha\beta}:E_\beta\to E_\alpha$ are induced by $\pi_{\alpha\beta}$. By Lemma 1, the conditions (i)–(iv) of [4, Theorem 1, p. 199] are satisfied (here $\mathfrak{S}_\alpha$ is the family of all linear subsets of E_α together with $\varnothing$). Therefore $\operatorname{proj\,lim} E_\alpha$ is nonempty. Let $z\in\operatorname{proj\,lim} E_\alpha$, then we have $\tau(x)=x$, i.e., τ is onto.

REMARK. This proof also works for inverse systems of rings as well as (noncommutative) groups.

COROLLARY 1. *If* $\{A_\alpha, \pi_{\alpha\beta}\}$ *is an inverse system of divisible Abelian groups satisfying the conditions: For every positive integer* n, (a) *each* $A_\alpha[n]$ *has a compact topology,* (b) *each* $\pi_{\alpha\beta}$ *induces a continuous homomorphism* $\pi_{\alpha\beta}[n]:A_\beta[n]\to A_\alpha[n]$, *then* $\operatorname{proj\,lim} A_\alpha$ *is also divisible.*

PROOF. Given a positive integer n, we have an exact sequence

$$0 \longrightarrow \{A_\alpha[n], \pi_{\alpha\beta}[n]\} \longrightarrow \{A_\alpha, \pi_{\alpha\beta}\} \xrightarrow{\{\tau_\alpha\}} \{A_\alpha, \pi_{\alpha\beta}\} \longrightarrow 0$$

where $\tau_\alpha(x)=nx$ for all $x\in A_\alpha$. By Lemma 2,

$$\tau = \operatorname{proj\,lim}\tau_\alpha:\operatorname{proj\,lim} A_\alpha \to \operatorname{proj\,lim} A_\alpha$$

is onto. We can verify directly that $\tau(x)=nx$ for all $x \in \text{proj lim } A_\alpha$. Therefore proj lim A_α is divisible.

REMARK. For bounded Abelian groups compactness coincides with linear compactness. There is no gain of generality to assume that the $A_\alpha[n]$ are linearly compact instead of being compact.

COROLLARY 2. *If* $\{A_\alpha, \pi_{\alpha\beta}\}$ *is an inverse system of Abelian groups where each* A_α *is a finite direct sum of quasi-cyclic groups then* proj lim A_α *is divisible.*

THEOREM 1. *If* $\{A_\alpha, \pi_{\alpha\beta}\}$ *is an inverse system of Abelian groups where* A_α *are finite direct sums of quasi-cyclic and bounded Abelian groups, then*

$$A_d = \text{proj lim}(A_\alpha)_d, \qquad A_r = \text{proj lim}(A_\alpha)_r.$$

As a consequence A is algebraically compact.

PROOF. We have an exact sequence of inverse systems of Abelian groups $0\to\{(A_\alpha)_d, \pi'_{\alpha\beta}\}\to\{A_\alpha, \pi_{\alpha\beta}\}\to\{(A_\alpha)_r, \pi''_{\alpha\beta}\}\to 0$ where $\pi'_{\alpha\beta}, \pi''_{\alpha\beta}$ are homomorphisms induced by $\pi_{\alpha\beta}$. By Lemma 2, the limit sequence

$$0 \to \text{proj lim}(A_\alpha)_d \to \text{proj lim } A_\alpha \to \text{proj lim}(A_\alpha)_r \to 0$$

is exact. By Corollary 2, proj $\lim(A_\alpha)_d$ is divisible. By [1, Proposition 39.4], proj $\lim(A_\alpha)_r$ is reduced. Therefore proj $\lim(A_\alpha)_d=A_d$, proj $\lim(A_\alpha)_r=A_r$.

LEMMA 3. *If* $\{A_\alpha, \pi_{\alpha\beta}\}$ *is an inverse system of torsion Abelian groups in which each* A_α *has only a finite number of nonzero primary components, then*

$$\text{proj lim } A_\alpha = \prod_p (\text{proj lim } T_p(A_\alpha)).$$

This is a consequence of the universal property of inverse limit.

LEMMA 4. *Let A be an Abelian group. A is the inverse limit of Abelian groups each of which is the direct sum of finite copies of* $Z(p^\infty)$ *iff* $A=\text{Hom}_Z(B, Z(p^\infty))$ *where B is a torsion-free Abelian group.*

PROOF. Let $\{A_\alpha, \pi_{\alpha\beta}\}$ be an inverse system of Abelian groups, where the A_α are finite direct sums of $Z(p^\infty)$, and $A=\text{proj lim } A_\alpha$.

Case I. All $\pi_{\alpha\beta}$ are onto. We have a direct system $\{\hat{A}_\alpha, \hat{\pi}_{\alpha\beta}\}$ of Abelian groups with

$$\hat{A}_\alpha = \text{Hom}_Z(A_\alpha, Z(p^\infty)), \qquad \hat{\pi}_{\alpha\beta} = \text{Hom}_Z(\pi_{\alpha\beta}, Z(p^\infty)).$$

Obviously $\hat{A}_\alpha$ are finite direct sums of $\hat{Z}_p$ and $\hat{\pi}_{\alpha\beta}$ are monomorphisms. We also have

$$A_\alpha = \text{Hom}_Z(\hat{A}_\alpha, Z(p^\infty)), \qquad \pi_{\alpha\beta} = \text{Hom}_Z(\hat{\pi}_{\alpha\beta}, Z(p^\infty)).$$

(These can be obtained either by direct computation or by Pontrjagin duality.) Therefore

$$\text{proj lim } A_\alpha = \text{proj lim } \mathrm{Hom}_Z(\hat{A}_\alpha, Z(p^\infty)) = \mathrm{Hom}_Z(\text{inj lim } \hat{A}_\alpha, Z(p^\infty)).$$

Since $\hat{A}_\alpha$ are torsion-free and $\hat{\pi}_{\alpha\beta}$ are 1-1, $B = \text{inj lim } \hat{A}_\alpha$ is torsion-free.

Case II (*the general case*). Let $\pi_\alpha : A \to A_\alpha$ be the inverse limit projections, $A'_\alpha = \mathrm{Im}\, \pi_\alpha$, and $\pi'_{\alpha\beta} : A'_\beta \to A'_\alpha$ be induced by $\pi_{\alpha\beta}$, then $\{A'_\alpha, \pi'_{\alpha\beta}\}$ is an inverse system. Obviously the $\pi'_{\alpha\beta}$ are onto, and $A = \text{proj lim } A'_\alpha$. By Case I, $A = \mathrm{Hom}_Z(B, Z(p^\infty))$, where B is a torsion-free Abelian group.

The converse is obvious.

COROLLARY. *An Abelian group is the inverse limit of finite direct sums of* $Z(p^\infty)$ *iff it is the direct product of copies of* $Z(p^\infty)$ *and copies of* Q *where the number of* Q *among the factors is either* 0 *or an infinite cardinal.*

This is a consequence of [**1**, Theorem 47.1].

Combining all the previous results we have

THEOREM 2. *An Abelian group* A *is the inverse limit of Abelian groups each of which is a finite direct sum of quasi-cyclic and bounded Abelian groups iff the following conditions are satisfied*:

(a) A_r *is complete.*

(b) $A_d = \prod_p \mathrm{Hom}_Z(B_p, Z(p^\infty))$, *where the* B_p *are torsion-free Abelian groups.*

COROLLARY. *Condition* (b) *can be replaced by*

(b′) A_d *is a direct product of quasi-cyclic groups and copies of* Q *where the number of* Q *among the factors is either* 0 *or an infinite cardinal.*

REMARK. Our results can be easily extended to modules over a Dedekind domain.

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