THE POSET OF SKEW FIELDS GENERATED BY A FREE ALGEBRA

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ABSTRACT. This paper shows that the poset of skew fields generated by a free algebra contains a subposet isomorphic to the lattice of all subsets of an infinite set.

Let R be a ring with identity. An R-field (K, λ) is a skew field K with an identity preserving ring homomorphism λ from R into K. If λ is understood from context, (K, λ) will be written as K. The R-field (K, λ) is an epic R-field if K is the smallest sub skew field of K containing $\lambda(R)$. Call two epic R-fields (K_1, λ_1) and (K_2, λ_2) equivalent if there exists an isomorphism γ from K_1 onto K_2 such that $\gamma\lambda_1 = \lambda_2$. By abuse of notation, epic R-field will refer to an equivalence class of epic R-fields. For epic R-fields (K_1, λ_1) and (K_2, λ_2) define $(K_1, \lambda_1) \geqq (K_2, \lambda_2)$ if there exists a local subring L of K_1 with $L \supseteq \lambda_1(R)$, and a homomorphism γ from L onto K_2 such that $\gamma\lambda_1 = \lambda_2$. A straightforward extension of [4, Lemma 1] shows that $\geqq$ is a partial ordering on the set of epic R-fields. This partially ordered set will be denoted by $\mathscr{F}_R$.

The set of epic R-fields (K, λ) with λ a one-to-one map is a subposet to be denoted by $\mathscr{G}_R$. For a commutative ring, $\mathscr{G}_R$ either contains one element or is empty depending on whether R is or is not an integral domain. For the ring R of polynomials in noncommuting variables with coefficients in the skew field k (hereafter called a free k-algebra), the structure is much richer. Moufang [9] in 1937 proved that $\mathscr{G}_R$ is not empty. Neumann [10] in 1949 proved that $|\mathscr{G}_R| \geqq 2$. Amitsur [1] in 1966 proved that $\mathscr{G}_R$ has a unique maximal element. Fisher [4] in 1971 proved that $\mathscr{G}_R$ contains infinitely many minimal elements. Cohn [2] gave a characterization for general rings in terms of sets of matrices over R called prime matrix ideals. Fisher [5] gave a characterization for general algebras R over infinite fields in terms of right Ore R-domains and homomorphisms.

This paper shows the following: Let F be the free group on the set X where the cardinality of X is at least two. Define $F'' = [[F, F], [F, F]]$. Let k be a skew field and R the free k-algebra on the set X.

THEOREM 1. $\mathscr{G}_R$ *contains a subposet corresponding to the set of normal subgroups H of F with $H \subseteq F''$ and F/H orderable, where $H_1 \geqq H_2$ if $H_2 \supseteq H_1$ and H_2/H_1 is a convex normal subgroup of F/H_1 under some ordering of F/H_1.*

THEOREM 2. $\mathscr{G}_R$ *contains a subposet isomorphic to the lattice of all subsets of an infinite set.*

In order to prove these theorems, we will use formal power series ([**10**] or see [**6**]). Let G be an ordered group and k a skew field. If ϕ is a map from G into k, then $\operatorname{supp}(\phi) = \{g \in G : \phi(g) \neq 0\}$. Define $D(G) = \{\phi : \operatorname{supp}(\phi)$ is well ordered in the full ordering of $G\}$. Addition and multiplication in $D(G)$ are given by

$$(\phi + \psi)(g) = \phi(g) + \psi(g), \qquad (\phi\psi)(g) = \sum_{hl=g} \phi(h)\psi(l).$$

It is proven in [**10**] that there is only a finite number of nonzero terms in the summation, and that $D(G)$ is a skew field. Furthermore $O(G) = \{\phi \in D(G) : g \geqq e$ for all $g \in \operatorname{supp}(\phi)\}$ is a valuation ring with corresponding valuation $v(\phi) = \min\{\operatorname{supp}(\phi)\}$ [**11**, p. 24]. The invariant primes of the associated valuation ring $O(G)$ correspond exactly to the convex normal subgroups of G [**11**, p. 14]. Furthermore, if H is a normal convex subgroup of G with corresponding invariant prime P then $O(G)\backslash P$ forms a right Ore system [**11**, p. 15] and the localization $S(G, H)$ maps naturally onto $D(G/H)$. Define $K(G)$ to be the smallest sub skew field of $D(G)$ containing the group ring kG. Let G_1 and G_2 be ordered groups which are isomorphic under an isomorphism α as groups (but not necessarily as ordered groups). Let λ_i be the natural isomorphisms of the group rings kG_i into $K(G_i)$, $i = 1, 2$. Hughes [**8**] has shown that there exists an isomorphism β of $K(G_1)$ onto $K(G_2)$ such that $\beta\lambda_1 = \lambda_2\alpha$. Hence with an orderable group G we associate the (unique up to kG-preserving isomorphism) skew field $K(G)$. Thus for a normal subgroup H which is convex under some ordering of G, $S(G, H) \cap K(G)$ is a local subring of $K(G)$ which maps homomorphically onto $K(G/H)$. Hence $K(G) \geqq K(G/H)$.

The proof of Theorem 1 will now be completed with the following lemma.

LEMMA 3. *The free semigroup on X is embedded in F/F''.*

PROOF. The proof follows easily from the matrix representation of F/F'' contained in [**7**].

Thus the free k-algebra on X is a subalgebra of kF/H where H is a normal subgroup of F, and $H \subseteq F''$. If in addition F/H is orderable then $K(F/H) \in \mathscr{G}_R$. It is clear that for H_1 and H_2 distinct normal subgroups of F with $H_i \subseteq F''$, $K(F/H_1)$ is not isomorphic to $K(F/H_2)$ as R-fields although it is entirely possible they may be isomorphic as skew fields. Hence Theorem 1 is proven.

PROOF OF THEOREM 2.   By [7] there exists a representation of $F/[F'', F]$ onto a group M of matrices. The kernel of the representation is a subgroup properly contained in $F''/[F'', F]$. M'' is central and free abelian on an infinite number of generators. Hence each subset S of the free generator set Y of M'' generates a central subgroup $H(S)$ of M. The group $M/H(S)$ is orderable since $F/F'' \cong M/M''$ is orderable and $M''/H(S)$ is central free abelian, isomorphic to $H(Y \setminus S)$. If $S_1 \subset S_2$ then $H(S_1) \subset H(S_2)$ and $H(S_2)/H(S_1)$ is a convex normal subgroup under an appropriate ordering of $M/H(S_1)$. Thus, by Theorem 1, $K(M/H(S_1)) \geqq K(M/H(S_2))$ and $\mathscr{G}_R$ contains a poset isomorphic to the lattice of subsets of Y.

COROLLARY. $\mathscr{G}_R$ contains infinite ascending and descending chains.

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