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ON RESTRICTED WEAK TYPE (1, 1)[1]

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ABSTRACT. Let $\{S_k\}_{k\geqq 1}$ be a sequence of linear operators defined on $L^1(R^n)$ such that for every $f\in L^1(R^n)$, $S_kf=f*g_k$ for some $g_k\in L^1(R^n)$, $k=1,2,\cdots$, and $Tf(x)=\sup_{k\geqq 1}|S_kf(x)|$. Then the inequality $m\{x\in R^n; Tf(x)>y\}\leqq Cy^{-1}\int_{R^n}|f(t)|\,dt$ holds for characteristic functions f (T is of restricted weak type (1, 1)) if and only if it holds for all functions $f\in L^1(R^n)$ (T is of weak type (1, 1)). In particular, if S_kf is the kth partial sum of Fourier series of f, this theorem implies that the maximal operator T related to S_k is not of restricted weak type (1, 1).

1. **Introduction.** We will show that maximal operators of a certain type are of weak type (1, 1) if and only if they are of restricted weak type (1, 1). Many important operators are of the type considered.

Throughout, R^n will denote n-dimensional Euclidean space, m will denote Lebesgue measure on R^n, and f will denote a measurable function on R^n. Recall that $L^p(R^n)$ is the set of all real (or complex) valued measurable functions on R^n with the property

$$\|f\|_p=\left(\int_{R^n}|f(x)|^p\,dx\right)^{1/p}<\infty,\qquad 1\leqq p<\infty,\tag{1.1}$$
$$\|f\|_\infty=\inf\{y;\, m\{x\in R^n:|f(x)|>y\}=0\}<\infty.$$

$C_c(R^n)$ will denote the set of all continuous functions on R^n with compact supports and $S(R^n)$ will denote the set of all simple functions each of which is a finite linear combination of characteristic functions of compact connected sets.

The convolution of measurable functions f and g on R^n is defined by

$$(f*g)(x)=\int_{R^n}f(t)g(x-t)\,dt\tag{1.2}$$

whenever the integral exists. Note that

$$\|f*g\|_1\leqq\|f\|_1\cdot\|g\|_1.\tag{1.3}$$

Let T be an operator defined on $L^p(R^n)$.

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T is of *weak type* (p, q) if there exists a positive constant A such that for each function f in $L^p(R^n)$ and $y>0$

$$m\{x \in R^n; |Tf(x)| > y\} \leqq ((A/y)\,\|f\|_p)^q. \tag{1.4}$$

T is of *restricted weak type* (p, q) if inequality (1.4) holds whenever f is restricted to the collection of characteristic functions of measurable set in R^n with finite measure.

It is obvious that T is of restricted weak type (p, q) if it is of weak type (p, q). But the converse is not true for $p>1$ (see [5]). We will, however, prove that for some special operators the converse is true for $p=1$.

2. **Restricted weak type** (p, q). Stein and Weiss [5] considered the operator T defined by

$$Tf(x) = x^{-1/q}\int_0^\infty y^{-1/p'}f(y)\,dy$$

and showed that T is of restricted weak type (p, q) but not of weak type (p, q), in the case $p>1$, where $1/p+1/p'=1$.

However, we are able to prove the following theorem:

THEOREM. *Let* S_n $(n=1, 2, \cdots)$ *be linear operators on* $L^1(R^m)$, *each of the form* $S_n f=f*g_n$ *for some* $g_n \in L^1(R^m)$, *and let* $Tf(x)=\sup_{n\geqq 1}|S_n f(x)|$.

Then, T *is of restricted weak type* $(1, q)$, $q\geqq 1$, *if and only if* T *is of weak type* $(1, q)$.

PROOF. It is enough to show that T is of weak type $(1, q)$ if it is of restricted weak type $(1, q)$ since the converse is trivial.

Let $f\geqq 0$ be a function in $S(R^m)$ such that $\|f\|_\infty\neq 0$. Since $C_c(R^m)$ is dense in $L^1(R^m)$, for any given $\varepsilon>0$, there exist $h_n \in C_c(R^m)$ $(n=1, 2, \cdots)$ such that

$$\|g_n - h_n\|_1 < \varepsilon/2\max(1, \|f\|_\infty). \tag{2.1}$$

Then we have

$$\begin{aligned} |f*g_n(x) - f*h_n(x)| &\leqq \int_{R^m} |f(t)|\,|g_n(x-t) - h_n(x-t)|\,dt \\ &\leqq \|f\|_\infty\,\|g_n - h_n\|_1 < \varepsilon/2. \end{aligned} \tag{2.2}$$

For any fixed $\lambda>0$ and all positive integers n, $1\leqq n\leqq N$, there exists $\delta=\delta(N)>0$ such that, for any connected set I with

$$\mathrm{dia}(I) = \sup\{|x-y|;\, x, y \in I\} < \delta,$$

$x, y \in I$ implies

$$|h_n(x) - h_n(y)| < \lambda/2\,\|f\|_1. \tag{2.3}$$

We now divide R^m into disjoint connected sets I_k such that $\mathrm{dia}(I_k)<\delta$ and $f(x)=\alpha_k$ on I_k where α_k's are positive real numbers. Note that such of α_k's are finitely many since $f \in S(R^m)$. Put $\alpha=\max\{\alpha_k\}$. Clearly $\alpha=\|f\|_\infty$.

Let F_k be a subinterval of I_k such that $m(F_k)=(\alpha_k/\alpha)m(I_k)$ and set $E_N=\bigcup_k F_k$. Thus, we have

$$\alpha m(E_N)=\sum_k \alpha m(F_k)=\sum_k \alpha_k m(I_k)=\|f\|_1. \tag{2.4}$$

Combining with (2.3) and applying the mean values theorem, we obtain, for each n, $1\leqq n\leqq N$,

$$\begin{aligned}|f*h_n(x)-\alpha\chi_{E_N}*h_n(x)| &= \left|\int_{R^m} f(t)h_n(x-t)\,dt-\alpha\int_{E_N}h_n(x-t)\,dt\right|\\ &\leqq \sum_k\left|\alpha_k\int_{I_k}h_n(x-t)\,dt-\alpha\int_{F_k}h_n(x-t)\,dt\right|\\ &= \sum_k |\alpha_k m(I_k)h_n(x-t_k)-\alpha m(F_k)h_n(x-t_k')| \\ &\qquad\qquad (\text{for some } t_k\in I_k \text{ and } t_k'\in F_k)\\ &= \sum_k \alpha m(F_k)\,|h_n(x-t_k)-h_n(x-t_k')|\\ &\leqq \sum_k \alpha m(F_k)\frac{\lambda}{2\,\|f\|_1}=\frac{\lambda}{2}.\end{aligned} \tag{2.5}$$

A combination of (2.2), (2.5), and (2.1) with $\alpha=\|f\|_\infty$ gives, for each n, $1\leqq n\leqq N$,

$$\begin{aligned}|S_nf(x)-\alpha S_n\chi_{E_N}(x)| &\leqq |f*g_n(x)-f*h_n(x)|\\ &\quad+|f*h_n(x)-\alpha\chi_{E_N}*h_n(x)|\\ &\quad+\alpha\,|\chi_{E_N}*h_n(x)-\chi_{E_N}*g_n(x)|\\ &\leqq \lambda/2+\varepsilon.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}T_Nf(x)=\sup_{1\leqq n\leqq N}|S_nf(x)| &\leqq \alpha T_N\chi_{E_N}(x)+\lambda/2+\varepsilon\\ &\leqq \alpha T\chi_{E_N}(x)+\lambda/2+\varepsilon.\end{aligned} \tag{2.6}$$

From (2.4) and the fact that T is of restricted weak type $(1,q)$, (2.6) implies

$$\begin{aligned}m\{x\in R^m;\ T_Nf(x)>\lambda+\varepsilon\} &\leqq m\{x\in R^m;\ T\chi_{E_N}(x)>\lambda/2\alpha\}\\ &\leqq \{(A/\lambda)\alpha m(E_N)\}^q=((A/\lambda)\,\|f\|_1)^q.\end{aligned}$$

Since $T_Nf(x)\leqq T_{N+1}f(x)$ for all $x\in R^m$ and $\varepsilon>0$ is arbitrary, we finally get

$$\begin{aligned}m\{x\in R^m;\ Tf(x)>\lambda\} &= \lim_{N\to\infty} m\{x\in R^m;\ T_Nf(x)>\lambda\}\\ &\leqq ((A/\lambda)\,\|f\|_1)^q \quad \text{for all } f\in S(R^m).\end{aligned} \tag{2.7}$$

We now consider a general function f in $L^1(R^m)$. Let N be a fixed positive integer. For any given $\varepsilon>0$, there exists a function $h_N\in S(R^m)$

such that

$$\|f-h_N\|_1 < \varepsilon^2/\max(1, M) \tag{2.8}$$

where $M=\max_{1\le n\le N}\|g_n\|_1$. Then, for each n, $1\leqq n\leqq N$, we have

$$\|S_nf-S_nh_N\|_1 \leqq \|g_n\|_1\,\|f-h_N\|_1 < \varepsilon^2$$

and

$$m\{x\in R^m;\ |S_nf(x)-S_nh_N(x)|>\varepsilon\} < \varepsilon. \tag{2.9}$$

Denote $B_n(N)=\{x\in R^m;\ |S_nf(x)-S_nh_N(x)|>\varepsilon\}$ and $B_N=\bigcup_{n=1}^N B_n(N)$. Then, for all $x\notin B_N$ and $n=1, 2, \cdots, N$,

$$T_Nf(x)=\sup_{1\leqq n\leqq N}|S_nf(x)| \leqq T_Nh_N(x)+\varepsilon \leqq Th_N(x)+\varepsilon.$$

From (2.7), (2.8), and (2.9), we get

$$\begin{aligned} m\{x\in R^m;\ T_Nf(x)>\lambda+\varepsilon\} &\leqq m\{x\in R^m;\ Th_N(x)>\lambda\}+mB_N \\ &\leqq \left(\frac{A}{\lambda}\,\|h_N\|_1\right)^q+\sum_{n=1}^N mB_n(N) \\ &\leqq \{(A/\lambda)(\|f\|_1+\varepsilon^2)\}^q+N\varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain

$$m\{x\in R^m;\ T_Nf(x)>\lambda\} \leqq ((A/\lambda)\,\|f\|_1)^q$$

and finally

$$m\{x\in R^m;\ Tf(x)>\lambda\}=\lim_{N\to\infty} m\{x\in R^m;\ T_nf(x)>\lambda\} \leqq \left(\frac{A}{\lambda}\,\|f\|_1\right)^q.$$

This completes the theorem.

3. **Applications.** Let $S_nf(x)$ be the nth partial sum of the Fourier series of $f(x)$ with respect to a complete orthonormal system $\{\phi_n;\ n=0, 1, 2, \cdots\}$ defined on a measurable set G in R, that is,

$$S_nf(x)=\sum_{j=0}^{n-1}\phi_j(x)\int_G f(t)\phi_j(t)\,dt \tag{3.1}$$

and let

$$Mf(x)=\sup_{n\geqq 1}|S_nf(x)|. \tag{3.2}$$

We will denote by $\Phi(L)$ the set of all measurable functions f on G such that

$$\int_G \Phi(|f(x)|)\,dx<\infty \tag{3.3}$$

and $\log^+ x=\max(0, \log x)$.

On the trigonometric system and the Walsh-Paley system, Sjölin [4] has shown that for each function f in the class $L(\log^+ L)(\log^+\log^+ L)$,

$S_n f(x)$ converges almost everywhere (a.e.) to $f(x)$ by using the fact that M is of restricted weak type (p, p), $1<p<\infty$ (so called "the basic result") ([2] and [4]). We also know that there exists a function f in the class $L(\log^+ \log^+ L)^{1-\varepsilon}$ for $\varepsilon>0$ such that $S_n f(x)$ diverges a.e. ([1] and [3] for the trigonometric system and [3] for the Walsh-Paley system).

The convergences or divergences of the functions in the classes between $L(\log^+ L)(\log^+ \log^+ L)$ and $L(\log^+ \log^+ L)$ for both systems are open questions.

Suppose that M were of restricted weak type $(1, 1)$. Then, by following the same proof of the a.e. convergence of functions in

$$L(\log^+ L)(\log^+ \log^+ L)$$

[4], we would be able to prove that for each function f in the class $L(\log^+ \log^+ L)$, $S_n f(x)$ converges a.e. to $f(x)$. But unfortunately we know that for both systems, M is not of weak type $(1, 1)$ and so is not of restricted weak type $(1, 1)$ by our theorem. This shows that the modification of the method in [2] and [4] to prove the almost everywhere convergence of functions in the class $L(\log^+ \log^+ L)$ is not available.

Let us note that the *maximal Hilbert transform* M defined by

$$Mf(x) = \sup_{n\geq 1} |H_n f(x)|, \tag{3.4}$$

where $H_n f(x)=\int_{1/n<|x-t|<n} f(t)/(x-t)\, dt$, is of the type that we have considered.

The *Hardy-Littlewood maximal operator* Λ defined by

$$\Lambda f(x) = \sup_{n\geq 1}\left(\frac{1}{|I_n(x)|}\int_{I_n(x)} |f(t)|\, dt\right), \tag{3.5}$$

where $I_n(x)$ is any interval with center at x and length 2^{-n} is essentially of this type.

References

1. Y. M. Chen, *An almost everywhere divergent Fourier series of the class* $L(\log^+ \log^+ L)^{1-\varepsilon}$, London Math. Soc. J. **44** (1969), 643–654. MR **39** #1886.

2. R. A. Hunt, *On the convergence of Fourier series*, Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press., Carbondale, Ill., 1968, pp. 235–255. MR **38** #6296.

3. K. H. Moon, *Divergent Fourier series of functions in Orlicz classes* (preprint in preparation).

4. P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Ark. Mat. 7 (1969), 551–570. MR **39** #3222.

5. E. M. Stein and G. Weiss, *An extension of a theorem of Marcinkiewicz and some of its applications*, J. Math. Mech. **8** (1959), 263–284. MR **21** #5888.

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