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SOLUTION OF A NONLINEAR PARTIAL DIFFERENTIAL EQUATION WITH INITIAL CONDITIONS

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ABSTRACT. The exact solution ϕ of a particular nonlinear partial differential equation is obtained in terms of solution u of a related linear partial differential equation. It is noted that solution ϕ may be found subject to initial conditions if certain initial conditions can be determined for solution u. Two examples are solved explicitly.

R. T. Herbst [1] has pointed out that the ordinary nonlinear differential equation

$$y'' + p(x)y' + kq(x)y = (1 - l)y'^2y^{-1} + \beta q(x)y^{1-l} \tag{1}$$

has the solution

$$y = [u + l\beta]^k, \qquad kl = 1, \beta = \text{const}, \tag{2}$$

provided that u satisfies the ordinary linear differential equation

$$u'' + p(x)u' + q(x)u = 0. \tag{3}$$

The purpose of this short note is to observe that (1) is readily generalized to the partial differential equation (6), below, in n independent variables $x=(x_1, \cdots, x_n)$.

To obtain this generalization, define the operator

$$L_k = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + kc(x); \tag{4}$$

substitute $\phi(x)$ defined by

$$\phi(x) = [u(x) + \beta l]^k \tag{5}$$

into the nonlinear differential equation

$$L_k\phi = f(x, \phi, \partial\phi/\partial x), \tag{6}$$

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where $f(x, \phi, \partial\phi/\partial x)$ represents the nonlinear terms to be determined; and, finally, make use of the assumption that $u(x)$ satisfies the linear equation

$$L_1 u = g(x), \tag{7}$$

where L_1 is (4) with $k=1$. The calculation thus amounts to carrying out the operations indicated by (6), this procedure providing an identity for f. The details are straightforward and are omitted.

Thus the nonlinear partial differential equation

$$L_k = (1-l)\phi^{-1}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} + [\beta c(x) + kg(x)]\phi^{1-l} \tag{8}$$

is satisfied by (5), provided $u(x)$ satisfies the linear partial differential equation (7). Let this linear equation be called the base equation. The function $g(x)$ is arbitrarily prescribed; its presence in (7) extends the ordinary differential equation of Herbst.

If, in a given problem, the nonlinear term ϕ^{1-l} is present while $c(x)\equiv 0$, it is obvious that the base equation must be nonhomogeneous. When $\beta c(x)\not\equiv 0$, the base equation may be homogeneous if the product $\beta c(x)$ can be adjusted, as is always the case if c is constant, to match a given coefficient of ϕ^{1-l}. The arbitrariness of β and $g(x)$ provides some flexibility in adapting (8) to a specified nonlinear equation.

An initial value problem can be posed for the solution $\phi(x)$ of the nonlinear equation (8) in terms of the solution $u(x)$ of the linear equation (7). Let x_0 denote initial values for any m of the n independent variables and let z denote the remaining $n-m$ variables, such that

$$\phi(z, x_0) = \phi_0(z) \tag{9}$$

represents a specified function of the x_0 initial values and z. The function

$$u(z, x_0) = u_0(z) \tag{10}$$

is to be determined, and, from (5), it clearly must be

$$u_0(z) = [\phi_0(z)]^l - \beta l, \tag{11}$$

where $\phi_0(z)\neq 0$ if $l<0$. Therefore if $u(x)$ satisfies the linear equation (7) and the initial condition (11), then $\phi(x)$ satisfies the nonlinear equation (8) and the initial condition (9).

Similarly, if an initial condition on any of the first derivatives of $\phi(x)$ is specified then a corresponding derivative of $u(x)$ may be determined. For example, suppose that

$$\partial\phi(x)/\partial x_i\big|_{x_i=x_{i0}} = F(z), \tag{12}$$

where $F(z)$ is some function specified at x_{i0}. It is thus required that

$$\partial u(x)/\partial x_i\big|_{x_i=x_{i0}} = l[\phi(z, x_{i0})]^{l-1}F(z), \tag{13}$$

where $\phi(z, x_{i0})\neq 0$ if $l<1$.

On the other hand, the specification of initial values for $u(x)$ and $\partial u(x)/\partial x_i$ imposes initial values on $\phi(x)$ and $\partial\phi(x)/\partial x_i$. Two explicit initial value problems are discussed below, after some special forms of (8) are noted.

For the special case $a_{ij}=0$, $i\neq j$, and $a_{ii}\equiv a_i$ constant, it is possible to put (8) in the form

$$\begin{aligned}\nabla_a^2\phi + \sum_{i=1}^{n} b_i(x)\frac{\partial\phi}{\partial x_i} + kc(x)\phi \\ = (1-l)\phi^{-1}(\nabla_a\phi)^2 + [\beta c(x) + kg(x)]\phi^{1-l},\end{aligned} \tag{14}$$

where $\sqrt{a_i}\partial/\partial x_i$ is the ith component of a slightly generalized gradient operator ∇_a such that

$$\nabla_a^2\phi = \sum_{i=1}^{n} a_i\frac{\partial^2\phi}{\partial x_i^2} \quad\text{and}\quad (\nabla_a\phi)^2 = \sum_{i=1}^{n} a_i\left(\frac{\partial\phi}{\partial x_i}\right)^2.$$

A nonlinear extension of the n-dimensional heat equation considered by Widder [2] follows from (14) in the form

$$\nabla_a^2\phi - \partial\phi/\partial t = (1-l)\phi^{-1}(\nabla_a\phi)^2 + g(x)\phi^{1-l}, \tag{15}$$

having a solution from the solution u of the base equation

$$\nabla_a^2 u - \partial u/\partial t = g(x), \tag{16}$$

where t denotes a time variable. In this case, the coefficients a_{ii} are all unity. The generalized gradient as defined above allows a nonlinear version of the equation studied by Lo [3] to take the same form as (15) but with $a_i=1$, $i=1, n-1$, say, and $a_n=\varepsilon$. A nonlinear Klein-Gordon equation would appear in this notation as

$$\nabla_a^2\phi + kM^2\phi = (1-l)\phi^{-1}(\nabla_a\phi)^2 + kg(x)\phi^{1-l}, \qquad \beta\equiv 0, \tag{17}$$

with $a_i=\sqrt{-1}$, $i=1,3$; $a_4=1$; and M=constant. A solution of (17) would follow from the nonhomogeneous Klein-Gordon equation

$$\nabla_a^2 u + M^2 u = g(x). \tag{18}$$

The idea of devising a solution of a nonlinear partial differential equation from that of a related linear partial differential equation has been applied by Montroll ([4], [5]) to models of population growth and diffusion. He has considered, among others, nonlinear equations of the form

$$D\nabla^2\phi - \partial\phi/\partial t = -D\{1 - [G'(\phi)/G(\phi)](\nabla\phi)^2\} + KG(\phi), \tag{19}$$

where D and K are constants. Montroll solved this equation for $G(\phi)=\phi(\theta-\phi)/\theta$, θ=constant, with several initial conditions. It is noted that (15) provides another type of nonlinear diffusion equation for which an exact solution is possible.

As an explicit example, consider the one-dimensional form of (15), i.e.,

$$D\phi_{xx} - \phi_t = (1 - l)\phi^{-1}D\phi_x^2 + g(x)\phi^{1-l}, \tag{20}$$

where x now denotes a single space variable defined in a closed interval $0 \leqq x \leqq L$. This one-dimensional equation is chosen for convenience to avoid undue complications. The problem is to solve (20) subject to the initial condition

$$\phi(x, 0) = \phi_0(x) = 0. \tag{21}$$

The condition that $u(x)$ must meet follows from (11) with $\beta=0$:

$$u(x, 0) = u_0(x) = [\phi_0(x)]^l = 0, \qquad l > 0. \tag{22}$$

Thus an initial value problem is possible if l is positive. A solution of the base equation (16) is known [6, p. 288] to be

$$u(x, t) = \sum_{m=1}^{\infty} \left\{\int_0^t g_m(\tau)\exp[-D(m\pi/L)^2(t - \tau)]d\tau\right\}\sin(m\pi x/L), \tag{23}$$

with

$$\begin{aligned} g(x, t) &= -\sum_{m=1}^{\infty} g_m(t)\sin(m\pi x/L), \\ g_m(t) &= \frac{2}{L}\int_0^L g(\xi, t)\sin(m\pi\xi/L)\, d\xi. \end{aligned} \tag{24}$$

Clearly, $u(x, 0)=0$ is possible from (23), and, hence, a solution of (20) and (21) is given by (23), the combination and (5). For this case, (20) is also satisfied for the boundary values

$$\phi(0, t) = 0, \qquad \phi(L, t) = 0, \qquad l > 0. \tag{25}$$

A second application of (15) is found in a solid state problem [7]. If the variables i and C are eliminated in equation (3) of [7], one obtains the ordinary differential equation

$$\phi'' + b\phi' = 2\phi^{-1}\phi'^2 - g\phi^2 \tag{26}$$

with constant coefficients, which is to be solved subject to the initial conditions

$$\phi(0) = 1, \qquad \phi'(0) = 0. \tag{27}$$

The solution of (26) is

$$\phi(t) = [u(t) - \beta]^{-1}, \tag{28}$$

the solution u being

$$u(t) = C_1 + C_2 \exp(-bt) + gt/b \tag{29}$$

with C_1 and C_2 arbitrary constants. Imposing condition (11), one obtains

$$u(0) = 1 + \beta, \qquad l = -1, \tag{30}$$

which also follows from (29) with $C_1=1$ and $C_2=\beta$. Imposing condition (13) on $u'(t)$, one must have $u'(0)=0$. Differentiation of (29) with C_1 and C_2 equal to 1 and β, respectively, shows that $u'(0)=0$ is secured if $\beta=g/b^2$. The initial value problem (26) and (27) is thus solved.

References

1. R. T. Herbst, *The equivalence of linear and nonlinear differential equations*, Proc. Amer. Math. Soc. **7** (1956), 95–97. MR **17**, 848.
2. D. V. Widder, *Series expansions of solutions of the heat equation in n dimensions*, Ann. Mat. Pura Appl. (4) **55** (1961), 389–409. MR **25** #331.
3. C. Y. Lo, *Series expansions of solutions of* $U_{xx}+U_{yy}+\varepsilon^2 U_{tt}=U_t$, SIAM J. Math. Anal. **3** (1972), 461–473.
4. E. W. Montroll, *On nonlinear processes involving population growth and diffusion*, J. Appl. Probability **4** (1967), 281–290. MR **36** #3585.
5. ———, *Lectures on nonlinear rate equations, especially those with quadratic nonlinearities*, Lectures in Theoretical Physics 10A: Quantum theory and statistical physics, Gordon and Breach, New York, 1968, pp. 531–573. MR **38** #4099a.
6. A. N. Tihonov and A. A. Samarskiĭ, *Equations of mathematical physics*, GITTL, Moscow, 1951; English transl., Pergamon, Oxford and Macmillan, New York, 1963. MR **16**, 130; **29** #2498.
7. A. C. Damask and G. J. Dienes, *Kinetics of vacancy-interstitial annihilation*. I. *Impurity-interstitial trapping*, Phys. Rev. **125** (1962), 444–446.

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