PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

DERIVATIONS OF AW^*-ALGEBRAS[1]

JAMES C. DEEL

ABSTRACT. It is proved that every derivation on an AW^*-algebra of type II_1 with central trace is inner. The proof employs a result on the algebraic decomposition of such algebras which is of interest even in the W^* case.

1. **Introduction and definitions.** In recent years the study of derivations on various kinds of C^*-algebras has received considerable attention. The most important result of this work is due to Sakai and Kadison: Every derivation on a W^*-algebra is inner. Much earlier, Kaplansky **[11]** showed that every derivation on an AW^*-algebra of type I is inner, so it is natural to conjecture that arbitrary AW^*-algebras also have this property. In Corollary 4.2 we will verify this for an AW^*-algebra of type II_1 with central trace.

Our proof employs a result on the algebraic decomposition of such algebras (Corollary 3.4) which shows that the algebraic decomposition is actually topological; the algebra is described as the set of bounded, weakly continuous fields of operators over the maximal ideal space. This result was discovered independently by Takemoto and Tomiyama **[14]** in the W^* case. The continuous theory is both simpler and more general than the usual measure-theoretic reduction for separably represented von Neumann algebras, and it appears likely that future refinements will provide a nonspatial reduction theory enjoying most of the features of the direct integral decomposition.

Let $\mathfrak{A}$ be a finite AW^*-algebra, $\mathfrak{Z}$ its center. F. B. Wright **[19]** showed that $\mathfrak{A}$ is strongly semisimple; i.e., $\mathfrak{A}$ can be imbedded as a subdirect product $\mathfrak{A}\subseteq \prod A_x$, $x\in X$, where each A_x is a simple C^*-algebra with identity 1_x, such that

(1) for each $x\in X$, $A_x=\{a(x):a\in\mathfrak{A}\}$.

Here X is the maximal ideal space of $\mathfrak{A}$, and $A_x\cong\mathfrak{A}/x$, $x\in X$. Furthermore, AW^*-algebras are weakly central **[19]**, and hence

(2) $\mathfrak{Z}$ consists precisely of the functions $x\mapsto\alpha(x)1_x$, $\alpha\in C(X)$.

Presented to the Society, August 7, 1972; received by the editors January 22, 1973 and, in revised form, April 11, 1973.

AMS (MOS) subject classifications (1970). Primary 46L05, 46L10.

Key words and phrases. Derivations, AW^*-algebras, algebraic reduction theory.

[1] This paper is based on a portion of the author's dissertation at the University of Kansas, supervised by Dr. T. W. Palmer.

This allows us to identify $\mathfrak{Z}$ with $C(X)$. Finally, if $\mathfrak{A}$ is of type II_1 with a central trace $\mathrm{Tr}:\mathfrak{A}\to\mathfrak{Z}$, then

(3) each A_x is a W^*-factor of type II_1, and

(4) for each $a \in \mathfrak{A}$, the map $x\mapsto \mathrm{tr}_x(a(x))$ is continuous on X, where $\mathrm{tr}_x: A_x\to C$ is the canonical trace of A_x.

Wright [19] proved that each A_x is an AW^*-factor of type II_1 having trace tr_x given by $\mathrm{tr}_x(a(x))=\mathrm{Tr}(a)(x)$, $a \in \mathfrak{A}$, and Feldman [4], [5] showed that any AW^*-factor with trace is a W^*-factor. In the type I case the quotients A_x are finite factors, but they may be of type II on a nowhere dense set in X. If one drops the assumption of a trace on $\mathfrak{A}$, then the A_x are finite AW^*-factors possibly without trace [21]; a proof of this is sketched in [1], along with an improved proof of strong semisimplicity. More recently, Takesaki [16] and Vesterstrom [17] have considered the quotient modulo an ideal which is not maximal.

It is not known whether all finite AW^*-algebras have central trace. There are rather weak conditions which imply the existence of a trace (see [2], [7], [20]), and Yen [21] showed that it would suffice to settle this question in the factor case. A finite AW^*-algebra $\mathfrak{A}$ has a central trace if and only if $\mathfrak{A}$ can be imbedded as an AW^*-subalgebra of a type I AW^*-algebra having the same center, and in this case $\mathfrak{A}$ is equal to its bicommutant in the type I algebra (see [7], [18]). J. Dixmier showed in [2] that finite W^*-algebras have central trace, and that if a C^*-algebra with identity has a central trace, then the trace is given by the approximation theorem, hence is unique.

If $\mathfrak{A}\subseteq\prod A_x$, $x \in X$, is the algebraic decomposition of a finite AW^*-algebra, then X is a Stonian space, since $\mathfrak{Z}$ is an AW^*-algebra. However, there are C^*-algebras which satisfy properties (1)–(4) above and have AW^* center, but which are not AW^*-algebras. For example, if X is a compact space and M is a W^*-factor of type II_1, then the C^* tensor product $C(X)\otimes M$ [15] is identified with the algebra of norm continuous functions from X into M, and hence this algebra satisfies (1)–(4); the same is true for the algebra of bounded, strong* continuous functions, or for any intermediate C^*-algebra. If X is a hyperstonian space [3] and M is a factor on a separable Hilbert space, then it follows from [5] that none of these examples is an AW^*-algebra.

Definition. Let X be a compact T_2 space, and for each $x \in X$ let A_x be a W^*-factor of type II_1. A C^*-subalgebra $\mathfrak{A}\subseteq\prod A_x$, $x \in X$, which satisfies conditions (1)–(4) above will be called a *type* II_1 *C^*-algebra with continuous trace*. In other words, $\mathfrak{A}$ is a strongly semisimple, weakly central C^*-algebra with identity which has a central trace, and whose simple components are factors of type II_1.

If one takes the finite factors A_x in this definition to be of type I instead of type II, then $\mathfrak{A}$ is an "ordinary" C^*-algebra with continuous trace; we will not pursue this analogy here, but it is responsible for our terminology. In the theory which follows there is really no need to exclude the possibility that some of the A_x are finite of type I. However, our results are of interest mainly in the type II case, since the structure of type I AW^*-algebras is known [**11**].

By examining the relationship between the algebraic decomposition of a type II_1 AW^*-algebra with trace and its representation as an algebra of module operators on an AW^*-module over the center, and then applying a theorem of H. Widom [**18**, Theorem 4.3], we will prove (Corollary 3.4) that if $\mathfrak{A}\subseteq\prod A_x$, $x \in X$, is a type II_1 C^*-algebra with continuous trace over a Stonian space X, then $\mathfrak{A}$ is an AW^*-algebra if and only if $\mathfrak{A}$ is "closed" in $\prod A_x$ in the following sense:

(5) If $b \in \prod A_x$, and if the map $x \mapsto \mathrm{tr}_x(a(x)b(x))$ is continuous on X for each $a \in \mathfrak{A}$, then $b \in \mathfrak{A}$.

The result on derivations is proved by showing that any derivation on an arbitrary type II_1 C^*-algebra with continuous trace is induced by an element b which satisfies the condition in (5).

Notation. If $\{V_i : i \in I\}$ is a collection of normed spaces, we denote by $\prod V_i$, $i \in I$, the set of functions $i \mapsto v(i) \in V_i$ such that

$$\sup\{\|v(i)\| : i \in I\} < \infty,$$

and we use the norm $\|v\| = \sup\|v(i)\|$, $i \in I$. If each V_i is a Banach space (resp. C^*-algebra, W^*-algebra), then so is $\prod V_i$, $i \in I$.

2. **Construction of the trace completion.** We begin with a simple lemma which will play a key role in this section and §3.

2.1 LEMMA. *Let X be a completely regular space, $\mathfrak{Z}=C(X)$, and for each $x \in X$ let V_x and W_x be Banach spaces. Let $\mathscr{V}$ be a linear subspace of $\prod V_x$, $x \in X$, satisfying*:

(1) *If $v \in \mathscr{V}$ and $\alpha \in \mathfrak{Z}$, then $\alpha v \in \mathscr{V}$, where $(\alpha v)(x) = \alpha(x)v(x)$, $x \in X$.*

(2) *For each $x \in X$, $\{v(x) : v \in \mathscr{V}\}$ is dense in V_x.*

(3) *For each $v \in \mathscr{V}$ the function $x \mapsto \|v(x)\|$ is upper semicontinuous.*

If $T: \mathscr{V} \to \prod W_x$, $x \in X$, is a bounded $\mathfrak{Z}$-linear map, then for each $x \in X$ there is a unique bounded operator $T_x: V_x \to W_x$ such that $T_x(v(x)) = (Tv)(x)$ for all $v \in \mathscr{V}$. Furthermore, $\sup\{\|T_x\| : x \in X\} = \|T\|$.

PROOF. Fix $x \in X$. For any $v \in \mathscr{V}$ and $\varepsilon > 0$ there is a neighborhood U of x such that $\|v(y)\| \leqq \|v(x)\| + \varepsilon$ for all $y \in U$. Let $\alpha: X \to [0, 1]$ be a continuous function with $\alpha(x) = 1$, $\alpha(X \backslash U) = \{0\}$. Then

$$\|\alpha v\| \leqq \sup\{\|v(y)\| : y \in U\} \leqq \|v(x)\| + \varepsilon,$$

so by $\mathfrak{Z}$-linearity of T,

$$\begin{aligned}\|(Tv)(x)\| &= \|\alpha(x)(Tv)(x)\| = \|(T(\alpha v))(x)\| \\ &\leqq \|T(\alpha v)\| \leqq \|T\| \cdot \|\alpha v\| \\ &\leqq \|T\|(\|v(x)\| + \varepsilon)\end{aligned}$$

for all $\varepsilon > 0$. Thus $\|(Tv)(x)\| \leqq \|T\| \cdot \|(v(x)\|$ for all $v \in \mathscr{V}$. Hence we can define T_x on the dense subspace $\{v(x): v \in \mathscr{V}\}$ by the desired formula, $T_x(v(x)) = (Tv)(x)$, and obtain a well-defined bounded linear map of norm at most $\|T\|$; let T_x also denote the unique extension of this map to all of V_x. Then $\|T_x\| \leqq \|T\|$. Finally, if $v \in \mathscr{V}$, then

$$\begin{aligned}\|Tv\| &= \sup\|(Tv)(x)\| = \sup\|T_x(v(x))\| \\ &\leqq \sup\|T_x\| \, \|v(x)\| \leqq (\sup\|T_x\|)\|v\|;\end{aligned}$$

hence $\sup\|T_x\| = \|T\|$. □

2.2 *Notation and discussion.* Let $\mathfrak{A} \subseteq \prod A_x$, $x \in X$, be a type II_1 C^*-algebra with continuous trace, $\mathfrak{Z} = C(X)$ its center. For each $a \in \mathfrak{A}$, define $\mathrm{Tr}(a) \in \mathfrak{Z}$ by $\mathrm{Tr}(a)(x) = \mathrm{tr}_x(a(x))$, $x \in X$; then Tr is a $\mathfrak{Z}$-valued trace on $\mathfrak{A}$, and hence defining $(a, b) = \mathrm{Tr}(b^*a)$, $a, b \in \mathfrak{A}$, provides a $\mathfrak{Z}$-valued inner product making $\mathfrak{A}$ into an inner product space over $\mathfrak{Z}$, as in [**18**]. Widom [**18**, p. 314] showed that any inner product space $\mathscr{M}$ over a commutative AW^*-algebra $\mathfrak{Z}$ can be completed to an AW^*-module $\mathscr{H}$ over $\mathfrak{Z}$, and it follows from [**18**, Lemma 2.4] that $\mathscr{H}$ is isomorphic to the $\mathfrak{Z}$-dual of $\mathscr{M}$, i.e., the set $\mathscr{M}^\dagger$ of all $\mathfrak{Z}$-linear maps $\omega: \mathscr{M} \to \mathfrak{Z}$ which are bounded relative to the norm defined by the inner product of $\mathscr{M}$. The goal of this section is to give an explicit description of $\mathfrak{A}^\dagger$ and, if X is a Stonian space, to show how the $\mathfrak{Z}$-valued inner product is defined on the completion of $\mathfrak{A}$.

For each $x \in X$, let H_x be the Hilbert space completion of A_x in the norm $\|a_x\|_2 = \mathrm{tr}_x(a_x^* a_x)^{1/2}$, $a_x \in A_x$. Since $\mathrm{tr}_x: A_x \to C$ has norm one, we have $\|a_x\|_2 \leqq \|a_x\|$, $a_x \in A_x$, and hence $\prod (A_x, \|\cdot\|) \subseteq \prod (A_x, \|\cdot\|_2) \subseteq \prod H_x$. The norm $\sup\|a(x)\|_2$, $x \in X$, on $\prod A_x$ will be denoted by $\|a\|_2$ when there is danger of confusion with the C^* norm, but we will omit the subscript on the norm of arbitrary elements of H_x or $\prod H_x$, $x \in X$.

Finally, define $\mathscr{H} = \{f \in \prod H_x : x \mapsto (a(x), f(x))$ is continuous on X for each $a \in \mathfrak{A}\}$. Clearly $\mathfrak{A} \subseteq \mathscr{H}$.

2.3 Proposition. *The $\mathfrak{Z}$-dual of $(\mathfrak{A}, \|\cdot\|_2)$ is $\mathfrak{A}^\dagger = \{\omega_f : f \in \mathscr{H}\}$, where $\omega_f(a) \in \mathfrak{Z}$ is defined by $\omega_f(a)(x) = (a(x), f(x))$, $a \in \mathfrak{A}$, $x \in X$. The correspondence $f \mapsto \omega_f$ is an isometric conjugate $\mathfrak{Z}$-linear isomorphism of $\mathscr{H}$ onto $\mathfrak{A}^\dagger$.*

Proof. Let $\omega \in \mathfrak{A}^\dagger$. By Lemma 2.1 with $V_x = H_x$, $W_x = C$, $\mathscr{V} = \mathfrak{A}$, we obtain, for each x, a bounded linear functional ω_x on H_x such that

$\omega(a)(x)=\omega_x(a(x))$, $a\in\mathfrak{A}$, and we have $\|\omega\|=\sup\|\omega_x\|$. Let $f(x)\in H_x$ be the vector such that $\omega_x(\cdot)=(\cdot,f(x))$. Then $\|f(x)\|=\|\omega_x\|$, so $f\in\prod H_x$ with $\|f\|=\|\omega\|$. For any $a\in\mathfrak{A}$, $(a(x),f(x))=\omega_x(a(x))=\omega(a)(x)$, which is a continuous function of x, since $\omega(a)\in\mathfrak{Z}$; hence $f\in\mathscr{H}$, and $\omega=\omega_f$. Conversely, if $f\in\mathscr{H}$, then $\omega_f:\mathfrak{A}\to\mathfrak{Z}$ is $\mathfrak{Z}$-linear, and $\|\omega_f(a)\|_{\mathfrak{Z}}=\sup|(a(x),f(x))|\leqq\|a\|_2\|f\|$; hence $\omega_f\in\mathfrak{A}^\dagger$. The last statement of the proposition follows by evaluating at each point. $\square$

2.4 PROPOSITION. *If $f\in\mathscr{H}$ and $x\mapsto\|f(x)\|$ is continuous at x_0, then $x\mapsto(f(x),g(x))$ is continuous at x_0 for any $g\in\mathscr{H}$.*

PROOF. Given $\varepsilon>0$, pick $a\in\mathfrak{A}$ with $\|a(x_0)-f(x_0)\|<\varepsilon/(3\|g\|)$. Then

$$\begin{aligned}
&|(f(x),g(x))-(f(x_0),g(x_0))|\\
&\quad\leqq|(f(x)-a(x),g(x))|\\
&\qquad+|(a(x),g(x))-(a(x_0),g(x_0))|+|(a(x_0)-f(x_0),g(x_0))|\\
&\quad\leqq\|f(x)-a(x)\|\cdot\|g\|+|(a(x),g(x))-(a(x_0),g(x_0))|+\varepsilon/3.
\end{aligned}$$

Since $\|f(x)-a(x)\|=[\|f(x)\|^2-2\operatorname{Re}(f(x),a(x))+\|a(x)\|^2]^{1/2}$ is continuous at x_0, and since $(a(x),g(x))$ is continuous, the first two terms will be less than $\varepsilon/3$ for all x close enough to x_0. $\square$

For future reference we note the following fact, which follows from the proof of 2.4.

2.5 COROLLARY. *Let $\mathscr{S}$ be a subset of $\mathfrak{A}$ such that for each $x\in X$, $\{s(x):s\in\mathscr{S}\}$ is dense in $(A_x,\|\cdot\|_2)$. If $g\in\prod H_x$ and $(s(x),g(x))$ is continuous on X for each $s\in\mathscr{S}$, then $g\in\mathscr{H}$.*

PROOF. To show that $(a(x),g(x))$ is continuous at x_0 if $a\in\mathfrak{A}$, approximate $a(x_0)$ by $s(x_0)$, where $s\in\mathscr{S}$, and proceed as in the proof of 2.4; $\|a(x)-s(x)\|_2$ is continuous since $a-s\in\mathfrak{A}$. $\square$

2.6 PROPOSITION. *If $f\in\mathscr{H}$, then the function $x\mapsto\|f(x)\|$ is lower semi-continuous.*

PROOF. We must show that the set $\{x\in X:\|f(x)\|\leqq c\}$ is closed for each $c\geqq0$. So let $\{x_i\}$ be a net in X with $\|f(x_i)\|\leqq c$ for all i, and let $x_i\to x$ in X. For any $a\in\mathfrak{A}$ we have

$$\begin{aligned}
c^2&\geqq\|f(x_i)\|^2-\|f(x_i)-a(x_i)\|^2\\
&=(a(x_i),f(x_i))+(f(x_i),a(x_i))-\|a(x_i)\|^2,
\end{aligned}$$

and the last expression is continuous, since $f\in\mathscr{H}$, so taking the limit as $x_i\to x$ gives $c^2\geqq\|f(x)\|^2-\|f(x)-a(x)\|^2$ for all $a\in\mathfrak{A}$. But A_x is dense in H_x, so $\|f(x)\|\leqq c$. $\square$

2.7 Proposition. *Let* $\mathfrak{A}$ *and* $\mathscr{H}$ *be as in* 2.2, *and assume now that* X *is a Stonian space. Then* $\mathscr{H}$ *is the completion of* $\mathfrak{A}$ *in Widom's sense. If* $f, g \in \mathscr{H}$, *then the function* $x \mapsto (f(x), g(x))$ *is continuous except on a set of the first category in* X, *and* $(f, g) \in \mathfrak{Z}$ *is the unique continuous function which coincides with* $(f(x), g(x))$ *at the points where the latter function is continuous. In particular,* $(f, g)(x) = (f(x), g(x))$ *if either* $\|f(\cdot)\|$ *or* $\|g(\cdot)\|$ *is continuous at* x, *by* 2.4.

Proof. The first statement follows from 2.3, as noted in 2.2. If $f, g \in \mathscr{H}$, then by 2.6 the functions

$$\alpha_1(x) = \|f(x) + g(x)\|^2, \qquad \alpha_2(x) = \|f(x) - g(x)\|^2,$$
$$\alpha_3(x) = \|f(x) + ig(x)\|^2, \qquad \alpha_4(x) = \|f(x) - ig(x)\|^2$$

are lower semicontinuous. By [3, p. 154] there exist $\beta_1, \cdots, \beta_4 \in \mathfrak{Z}$ with $\beta_i(x) = \alpha_i(x)$ except on a set of the first category. By the polarization identity, the continuous function $(f, g) = (\beta_1 - \beta_2 + i\beta_3 - i\beta_4)/4$ coincides with $(f(x), g(x))$ except on a set of the first category. By computing on a dense subset of X one easily verifies that this definition of (f, g) provides an inner product on $\mathscr{H}$ which extends that of $\mathfrak{A}$. The AW^*-module axioms [**11**, p. 842] can also be verified directly using 2.3, making this construction of $\mathscr{H}$ independent of Widom's construction. □

3. **The standard module representation.** Throughout this section let $\mathfrak{A} \subseteq \prod A_x$, $x \in X$, be a type II_1 C^*-algebra with continuous trace, and let $\mathscr{H} \subseteq \prod H_x$ be as in 2.2. Unless otherwise stated we will assume that X is a Stonian space, so that $\mathscr{H}$ is an AW^*-module over $\mathfrak{Z} = C(X)$, by 2.7. Let $\mathfrak{B}$ denote the set of all bounded $\mathfrak{Z}$-linear maps $T: \mathscr{H} \to \mathscr{H}$; $\mathfrak{B}$ is an AW^*-algebra of type I with center $\mathfrak{Z}$ [**11**]. For each $x \in X$, let B_x be the algebra of all bounded operators on H_x. We will study the representation of $\mathfrak{A}$ on $\mathscr{H}$ defined by left multiplication on the submodule $\mathfrak{A} \subseteq \mathscr{H}$, and identify the bicommutant of the image of $\mathfrak{A}$ in $\mathfrak{B}$.

3.1 Lemma. *If* $T \in \mathfrak{B}$, *then for each* $x \in X$ *there is a unique operator* $T_x \in B_x$ *such that* $(Ta)(x) = T_x a(x)$ *for all* $a \in \mathfrak{A}$, *and* $\|T\| = \sup \|T_x\|$, $x \in X$. *For each* x, *the map* $T \mapsto T_x$ *is linear and* $(T^*)_x = (T_x)^*$. *Finally, any* $\mathfrak{Z}$*-linear map* $T: \mathfrak{A} \to \mathscr{H}$ *which is bounded relative to the norm* $\|\cdot\|_2$ *on* $\mathfrak{A}$ *extends uniquely to an element of* $\mathfrak{B}$.

Proof. The first statement follows by taking $\mathscr{V} = \mathfrak{A}$ in Lemma 2.1. Linearity follows from uniqueness. For any $a, b \in \mathfrak{A}$,

$$\begin{aligned}(a(x), T_x^* b(x)) &= (T_x a(x), b(x)) = ((Ta)(x), b(x)) \\ &= (Ta, b)(x) = (a, T^* b)(x) = (a(x), (T^*)_x b(x)),\end{aligned}$$

by 2.7. Thus $(T_x)^*=(T^*)_x$, since A_x is dense in H_x. The last statement is [**18**, Lemma 2.4]. $\square$

Define $\overline{\mathfrak{A}}=\mathscr{H}\cap\prod A_x=\{b\in\prod A_x:\mathrm{tr}_x(a(x)b(x))$ is continuous on X for each $a\in\mathfrak{A}\}$, and note that $\overline{\mathfrak{A}}$ is a two-sided module over $\mathfrak{A}$, even if X is not Stonian. For if $a\in\mathfrak{A}$ and $b\in\overline{\mathfrak{A}}$, then for any $c\in\mathfrak{A}$, $\mathrm{tr}_x(c(x)a(x)b(x))$ is continuous since $ca\in\mathfrak{A}$, and $\mathrm{tr}_x(c(x)b(x)a(x))=\mathrm{tr}_x(a(x)c(x)b(x))$ is continuous since $ac\in\mathfrak{A}$; hence $ab\in\overline{\mathfrak{A}}$ and $ba\in\overline{\mathfrak{A}}$.

If $b\in\overline{\mathfrak{A}}$, then $\|ba\|_2\leqq\|b\|\cdot\|a\|_2$ for all $a\in\mathfrak{A}$, so by 3.1 the map $a\mapsto ba$ extends uniquely to an operator $L_b\in\mathfrak{B}$. Let $R_b\in\mathfrak{B}$ be the operator defined similarly using right multiplication. In the notation of 3.1, $(L_b)_x=L_{b(x)}\in B_x$, and hence $(L_b)^*=L_{b^*}$ and $\|L_b\|=\sup\|L_{b(x)}\|=\|b\|$; likewise for R_b.

3.2 Lemma. (i) *If* $b\in\mathfrak{A}$ *and* $f\in\mathscr{H}$, *then* $(L_bf)(x)=L_{b(x)}f(x)$ *and* $(R_bf)(x)=R_{b(x)}f(x)$ *for all* $x\in X$.

(ii) $L:\overline{\mathfrak{A}}\to\mathfrak{B}$ *is an* $\mathfrak{A}$*-linear*, $*$*-preserving isometry.*

Proof. (i) Let $b\in\mathfrak{A}, f\in\mathscr{H}$. For any $a\in\mathfrak{A}$ and any $x\in X$,

$$\begin{aligned}((L_bf)(x),a(x)) &= (L_bf,a)(x) = (f,b^*a)(x)\\ &= (f(x),L_{b^*(x)}a(x)) = (L_{b(x)}f(x),a(x)),\end{aligned}$$

by the last part of 2.7. Since A_x is dense in H_x, this proves the first formula; the formula for R_b is proved in the same way.

(ii) If $a\in\mathfrak{A}$, $b\in\overline{\mathfrak{A}}$, then for any $c\in\mathfrak{A}$ we have $L_aL_b(c)=L_a(bc)=abc=L_{ab}(c)$ by part (i), and $L_bL_a(c)=bac=L_{ba}(c)$ is trivial. Hence $L_aL_b=L_{ab}$ and $L_bL_a=L_{ba}$. $\square$

3.3 Theorem. *Let* $\mathfrak{A}\subseteq\prod A_x$, $x\in X$, *be a type* II_1 C^**-algebra with continuous trace*, X *Stonian. Let* $L_{\mathfrak{A}}\subseteq\mathfrak{B}$ *denote the image of* $\mathfrak{A}$ *under* L. *Then the bicommutant of* $L_{\mathfrak{A}}$ *in* $\mathfrak{B}$ *is* $L_{\overline{\mathfrak{A}}}$.

Proof. First we claim that $L'_{\mathfrak{A}}=R_{\overline{\mathfrak{A}}}$. If $a\in\mathfrak{A}$ and $b\in\overline{\mathfrak{A}}$, then $L_aR_b(c)=acb=R_bL_a(c)$ for any $c\in\mathfrak{A}$, by 3.2(i). Hence $L_aR_b=R_bL_a$, and so $R_{\overline{\mathfrak{A}}}\subseteq L'_{\mathfrak{A}}$. Conversely, if $T\in L'_{\mathfrak{A}}$ and $T_x\in B_x$ is as in 3.1, then for any $a,c\in\mathfrak{A}$ and any $x\in X$,

$$0=(TL_a-L_aT)(c)(x)=(T_xL_{a(x)}-L_{a(x)}T_x)(c(x))$$

by 3.2(i). Hence T_x belongs to the commutant of L_{A_x} in B_x, and this commutant is R_{A_x}. Thus $T_x=R_{b(x)}$ for some $b(x)\in A_x$, and $\sup\|b(x)\|=\sup\|T_x\|=\|T\|$ by 3.1, so $b\in\prod A_x$. For any $a\in\mathfrak{A}$, $(Ta)(x)=a(x)b(x)$, so taking $a=1$ gives $T(1)=b\in\mathscr{H}$. Thus $b\in\overline{\mathfrak{A}}$ and $T=R_b$ on $\mathfrak{A}$, hence on all of $\mathscr{H}$. So $L'_{\mathfrak{A}}=R_{\overline{\mathfrak{A}}}$.

Similar reasoning shows that $R'_{\mathfrak{A}}=L_{\overline{\mathfrak{A}}}$, and so $L''_{\mathfrak{A}}=(R_{\overline{\mathfrak{A}}})'\subseteq R'_{\mathfrak{A}}=L_{\overline{\mathfrak{A}}}$. Thus it remains to show that $L_{\overline{\mathfrak{A}}}\subseteq(R_{\overline{\mathfrak{A}}})'$. So let $a,b\in\overline{\mathfrak{A}}$; it suffices to show that $(L_aR_bc,d)=(R_bL_ac,d)$, all $c,d\in\mathfrak{A}$, i.e., $(cb,a^*d)=(ac,db^*)$. Except on a set of the first category in X we have $(cb, a^*d)(x)=(c(x)b(x), a^*(x)d(x))$ and $(ac, db^*)(x)=(a(x)c(x), d(x)b^*(x))$, by 2.7. Hence the continuous functions (cb, a^*d) and (ac, db^*) coincide on a dense set, so are equal. □

3.4 Corollary. *Let* $\mathfrak{A}$ *be as in the theorem. Then* $\mathfrak{A}$ *is an* AW^*-*algebra if and only if* $\mathfrak{A}=\tilde{\mathfrak{A}}$.

Proof. In general, $L_{\tilde{\mathfrak{A}}}$ is an AW^*-algebra, in fact an AW^*-subalgebra of the type I algebra $\mathfrak{B}$, since it is a commutant in $\mathfrak{B}$ by 3.3. If $\mathfrak{A}=\tilde{\mathfrak{A}}$, then $L:\mathfrak{A}\to\mathfrak{B}$ is an isomorphism of $\mathfrak{A}$ onto $L_{\tilde{\mathfrak{A}}}$, so $\mathfrak{A}$ is an AW^*-algebra. Conversely, if $\mathfrak{A}$ is an AW^*-algebra, then it must be finite, since it has a trace. Hence, by the proof of [**18**, Theorem 4.3], $L_{\mathfrak{A}}$ is equal to its bicommutant in $\mathfrak{B}$, and so $\mathfrak{A}=\tilde{\mathfrak{A}}$ by Theorem 3.3. □

3.5 Remarks. (1) If X is a hyperstonian space, i.e., $\mathfrak{Z}$ is a W^*-algebra [**3**], then the type I algebra $\mathfrak{B}$ is a W^*-algebra [**10**], and $L_{\tilde{\mathfrak{A}}}$ is weakly closed in any representation of $\mathfrak{B}$ as a von Neumann algebra, since it is a commutant. So in this case 3.4 determines when $\mathfrak{A}$ is a W^*-algebra.

(2) Even if $\mathfrak{A}\neq\tilde{\mathfrak{A}}$, the AW^*-algebra $L_{\tilde{\mathfrak{A}}}$ has center $L_{\mathfrak{Z}}$, and $L_{\tilde{\mathfrak{A}}}$ is finite since it has the obvious central trace. Given a Stonian space X and a type II_1 factor M, we obtain a type II_1 AW^*-algebra with trace by letting $\mathfrak{A}=C(X)\otimes M$ and forming $L_{\tilde{\mathfrak{A}}}$; this generalizes a construction of Yen [**20**]. In this case, $\tilde{\mathfrak{A}}$ is the set of bounded, weakly continuous functions from X into M. Feldman's results [**5**] show that the simple components $\tilde{A}_x$ of $L_{\tilde{\mathfrak{A}}}$ need not be isomorphic to M if M is a factor on a separable space. One can show that in the general case, A_x is imbedded in $\tilde{A}_x$, and that if $b\in\tilde{\mathfrak{A}}$, then $(L_b)(x)$ "belongs to" A_x except on the set of the first category where the function $\|b(\cdot)\|_2$ is discontinuous. If $a, b\in\tilde{\mathfrak{A}}$, then $L_aL_b=L_c$, where c is the unique element of $\tilde{\mathfrak{A}}$ such that $c(x)=a(x)b(x)$ except on a set of the first category.

(3) After Feldman's discovery [**5**] that the factors which occur in the algebraic decomposition of a W^*-algebra are in some ways more complicated than the algebra itself, there was little reason to prefer the algebraic approach. The direct integral gives a measure theoretic decomposition. of a separably represented von Neumann algebra into separably represented factors, and it is not restricted to the finite case. However, in Corollary 3.4 we have a description of the elements of $\prod A_x$ which belong to $\mathfrak{A}$ in terms of a topological condition which is easier to deal with than the corresponding measure theoretic description. And as remark (2) indicates, it should be possible to avoid the pathologies discovered by Feldman by working modulo sets of the first category; the idea would be to find a "well-behaved" subalgebra $\mathfrak{A}_0\subseteq\mathfrak{A}$ such that $\mathfrak{A}\simeq L_{\tilde{\mathfrak{A}}_0}$. We will

pursue these refinements, along with an examination of the semifinite case (see [13]), in a later paper.

(4) Takemoto and Tomiyama [14] have proved independently that $\mathfrak{A}=\overline{\mathfrak{A}}$ if $\mathfrak{A}$ is a finite, σ-finite W^*-algebra. They introduce a "generalized predual space" for $\mathfrak{A}$, and by using a normal measure on X, they obtain this result from Sakai's characterization of W^*-algebras. It appears that a similar proof of our result might be based on Halpern's generalization [9] of Sakai's theorem.

In their theory, Takemoto and Tomiyama include decompositions over the maximal ideal space of a W^*-subalgebra $\mathfrak{Z}_0$ of $\mathfrak{Z}$. The case $\mathfrak{Z}_0=\mathfrak{Z}$ provides the finest decomposition of $\mathfrak{A}$, and insures that the component W^*-algebras will be factors. However, the greater generality is useful in comparing the decomposition of $\mathfrak{A}$ with that of a subalgebra, and it allows them to incorporate the results of Takesaki [16] and Vesterstrom [17] into their theory. They also give an interesting characterization of the W^*-subalgebras of $\mathfrak{A}$ which contain $\mathfrak{Z}_0$ in terms of the existence of sufficiently many extreme points. In our approach, we would need to assume the existence of a suitable $\mathfrak{Z}_0$-valued state, where $\mathfrak{Z}_0$ is an AW^*-subalgebra of $\mathfrak{Z}$, and use [18, Theorem 4.4] to conclude that $\mathfrak{A}$ is equal to its bicommutant in the resulting imbedding in a type I algebra with center $\mathfrak{Z}_0$.

4. **Derivations.** In this section we return to the case of a type II_1 C^*-algebra with continuous trace $\mathfrak{A}\subseteq\prod A_x$, $x\in X$, over an arbitrary compact T_2 space X. As in §3, let $\overline{\mathfrak{A}}=\prod A_x\cap\mathscr{H}$, which forms a two sided module over $\mathfrak{A}$.

4.1 Theorem. *If δ is a derivation on $\mathfrak{A}$, then there exists $d\in\overline{\mathfrak{A}}$ such that $\delta(a)=da-ad$, $a\in\mathfrak{A}$.*

Proof. Since $\prod A_x$ is a W^*-algebra, there exists $d\in\prod A_x$ with $\delta(a)=da-ad$, $a\in\mathfrak{A}$, by Sakai's derivation theorem. Since d' defined by $d'(x)=d(x)-\mathrm{tr}_x(d(x))1_x$ also induces δ, we may assume to begin with that $\mathrm{tr}_x(d(x))=0$ for all $x\in X$. Using only the fact that $da-ad\in\overline{\mathfrak{A}}$ for all $a\in\mathfrak{A}$, we have, for any $a, b\in\mathfrak{A}$, that

$$\mathrm{tr}_x(b(x)[d(x)a(x)-a(x)d(x)])=\mathrm{tr}_x([a(x)b(x)-b(x)a(x)]d(x))$$

is continuous, i.e., $(c(x), d(x))$ is continuous for each commutator c of $\mathfrak{A}$. Let $\mathscr{S}$ be the linear span of $\{1\}\cup\{\text{commutators of } \mathfrak{A}\}$; then $(s(x), d(x))$ is continuous for each $s\in\mathscr{S}$. To show that $d\in\overline{\mathfrak{A}}$ it suffices, by 2.5, to show that $\{s(x): s\in\mathscr{S}\}$ is dense in $(A_x, \|\cdot\|_2)$ for each $x\in X$. The following argument shows that this set is even dense relative to the C^* norm of A_x.

Let A be a finite factor, tr its canonical trace. Let $a\in A$, and let $\varepsilon>0$.

By the approximation theorem [2] there exist unitary elements $u_1, \cdots, u_n \in A$ and real numbers $t_1, \cdots, t_n$, $t_i > 0$, $\sum t_i = 1$, such that

$$\|\mathrm{tr}(a) - \sum t_i u_i^* a u_i\| < \varepsilon.$$

Thus

$$\|a - \sum t_i[u_i(u_i^* a) - (u_i^* a)u_i] - \mathrm{tr}(a)\| < \varepsilon,$$

i.e., a can be approximated by a linear combination of commutators and scalars. □

4.2 COROLLARY. *If $\mathfrak{A}$ is a type II_1 AW^*-algebra with trace, then every derivation on $\mathfrak{A}$ is inner.*

PROOF. In this case $\mathfrak{A} = \bar{\mathfrak{A}}$ by Corollary 3.4.

4.3 REMARKS. (1) Theorem 4.1 remains valid if one assumes only that δ is a derivation from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$. This form of 4.1 can be expressed concisely in the language of cohomology theory by the statement $H^1(\mathfrak{A}, \bar{\mathfrak{A}}) = 0$. If $\delta : \mathfrak{A} \to \bar{\mathfrak{A}}$ is a derivation, then $\delta(\mathfrak{Z}) \subseteq \mathfrak{Z}$, hence $\delta|_{\mathfrak{Z}} = 0$, and so δ is $\mathfrak{Z}$-linear. A recent result of J. Ringrose [12, Theorem 2] shows that δ is norm continuous, and hence Lemma 2.1 provides a decomposition of δ into linear maps $\delta_x : A_x \to A_x$ such that $\delta(a)(x) = \delta_x(a(x))$, $a \in \mathfrak{A}$, $x \in X$, and $\sup\|\delta_x\| = \|\delta\|$. (The function $\|a(\cdot)\|$ is upper semicontinuous on X for each $a \in \mathfrak{A}$ by [6, Lemma 9].) Each δ_x is a derivation, so by Sakai's theorem there is an element $d(x) \in A_x$ which induces δ_x with $\|d(x)\| \leqq \|\delta_x\|$. Then $d \in \prod A_x$, and $\delta(a) = da - ad$, $a \in \mathfrak{A}$. The proof of 4.1 applies to show that d can be chosen in $\bar{\mathfrak{A}}$.

(2) An example of A. A. Hall [8] shows that a type II_1 C^*-algebra with continuous trace can have outer derivations. His example shows that this occurs if $\mathfrak{A} = C(X) \otimes M$, where $X = \{1, 2, \cdots, \infty\}$ and M is the hyperfinite factor of type II_1.

ACKNOWLEDGEMENT. I would like to thank Theodore W. Palmer for suggesting the derivation problem, and for many helpful discussions. I am also indebted to the referee for providing me with a copy of [14], and for suggesting several improvements in the exposition.

REFERENCES

1. S. K. Berberian, *On the projection geometry of a finite AW*-algebra*, Trans. Amer. Math. Soc. **83** (1956), 493–509. MR **19**, 47.

2. J. Dixmier, *Les anneaux d'opérateurs de classe fini*, Ann. Sci. École Norm. Sup. (3) **66** (1949), 209–261. MR **11**, 370.

3. ———, *Sur certains espaces considérées par M. H. Stone*, Summa Brasil. Math. **2** (1951), 151–182. MR **14**, 69.

4. J. Feldman, *Embedding of AW*-algebras*, Duke Math. J. **23** (1956), 303–307. MR **17**, 1229.

5. J. Feldman, *Nonseparability of certain finite factors*, Proc. Amer. Math. Soc. **7** (1956), 23–26. MR **18,** 139.

6. J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras*, Ann. of Math. (2) **72** (1960), 216–244. MR **22** #7005.

7. M. Goldman, *Structure of AW^*-algebras*. I, Duke Math. J. **23** (1956), 23–34. MR **17,** 512; 1437.

8. A. A. Hall, *Derivations of certain C^*-algebras*, J. London Math. Soc. (2) **5** (1972), 321–329.

9. H. Halpern, *Embedding as a double commutator in a type* I *AW^*-algebra*, Trans. Amer. Math. Soc. **148** (1970), 85–98. MR **41** #837.

10. I. Kaplansky, *Algebras of type* I, Ann. of Math. (2) **56** (1952), 460–472. MR **14,** 291.

11. ———, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858. MR **15,** 327.

12. J. R. Ringrose, *Automatic continuity of derivations of operator algebras*, J. London Math. Soc. (2) **5** (1972), 432–438.

13. S. Strătilă and L. Zsidó, *An algebraic reduction theory for W^*-algebras*. I, J. Functional Analysis **11** (1972), 295–313.

14. H. Takemoto and J. Tomiyama, *On the reduction of finite von Neumann algebras* (to appear).

15. M. Takesaki, *On the cross-norm of the direct product of C^*-algebras*, Tôhoku Math. J. (2) **16** (1964), 111–122. MR **29** #2668.

16. ———, *The quotient algebra of a finite von Neumann algebra*, Pacific J. Math. **36** (1971), 827–831. MR **43** #6739.

17. J. Vesterstrom, *Quotients of finite W^*-algebras*, J. Functional Analysis **9** (1972), 322–335.

18. H. Widom, *Embedding in algebras of type* I, Duke Math. J. **23** (1956), 309–324. MR **17,** 1228.

19. F. B. Wright, *A reduction for algebras of finite type*, Ann. of Math. (2) **60** (1954), 560–570. MR **16,** 375.

20. T. Yen, *Trace on finite AW^*-algebras*, Duke Math. J. **22** (1955), 207–222. MR **16,** 1033.

21. ———, *Quotient algebra of a finite AW^*-algebra*, Pacific J. Math. **6** (1956), 389–395. MR **18,** 138.

Department of Mathematics, University of Kansas, Lawrence, Kansas 66044

Current address: Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221