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MUTUAL EXISTENCE OF PRODUCT INTEGRALS

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ABSTRACT. Definitions and integrals are of the subdivision-refinement type, and functions are from $R\times R$ to R, where R represents the real numbers. Let OM° be the class of functions G such that ${}_x\prod^y(1+G)$ exists for $a\leqq x<y\leqq b$ and $\int_a^b|1+G-\prod(1+G)|=0$. Let OP° be the class of functions G such that $|\prod_{q=i}^{j}(1+G_q)|$ is bounded for refinements $\{x_q\}_{q=0}^n$ of a suitable subdivision of $[a,b]$. If F and G are functions from $R\times R$ to R such that $F\in OP^\circ$ on $[a,b]$, $\lim_{x,y\to p^+}F(x,y)$ and $\lim_{x,y\to p^-}F(x,y)$ exist and are zero for $p\in[a,b]$, each of $\lim_{x\to p^+}F(p,x)$, $\lim_{x\to p^-}F(x,p)$, $\lim_{x\to p^+}G(p,x)$ and $\lim_{x\to p^-}G(x,p)$ exist for $p\in[a,b]$, and G has bounded variation on $[a,b]$, then any two of the following statements imply the other: (1) $F+G\in OM^\circ$ on $[a,b]$, (2) $F\in OM^\circ$ on $[a,b]$, and (3) $G\in OM^\circ$ on $[a,b]$.

All integrals and definitions are of the subdivision-refinement type, and functions are from $R\times R$ to R, where R represents the set of real numbers. Furthermore, functions are assumed to be defined only for elements $\{x,y\}$ of $R\times R$ such that $x<y$. If $D=\{x_q\}_{q=0}^n$ is a subdivision of $[a,b]$, then $D(I)=\{[x_{q-1},x_q]\}_{q=1}^n$ and $G_q=G(x_{q-1},x_q)$. Further, $\{x_{qr}\}_{r=0}^{n(q)}$ represents a subdivision of the interval $[x_{q-1},x_q]$ and $G_{qr}=G(x_{q,r-1},x_{qr})$. The statement that $\int_a^b G$ exists means there exists a number L such that, if $\varepsilon>0$, then there exists a subdivision D of $[a,b]$ such that if J is a refinement of D, then

$$\left|L-\sum_{J(I)}G\right|<\varepsilon.$$

The statement that ${}_a\prod^b(1+G)$ exists means there exists a number L such that, if $\varepsilon>0$, then there exists a subdivision D of $[a,b]$ such that if J is a refinement of D, then

$$\left|L-\prod_{J(I)}(1+G)\right|<\varepsilon.$$

Further, $G\in OA^\circ$ on $[a,b]$ only if $\int_a^b G$ exists and $\int_a^b|G-\int G|=0$, and

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$G \in OM^{\circ}$ on $[a, b]$ only if ${}_x\prod^y (1+G)$ exists for $a \leqq x < y \leqq b$ and $\int_a^b |1+G-\prod (1+G)|=0$.

The statements that G is bounded, $G \in OP^{\circ}$, $G \in OQ^{\circ}$ and $G \in OB^{\circ}$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and positive numbers B and β such that if $J=\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|G(u)|<B$ for $u \in J(I)$,

(2) $|\prod_{q=r}^s (1+G_q)|<B$ for $1 \leqq r \leqq s \leqq n$,

(3) $|\prod_{q=r}^s (1+G_q)|>\beta$ for $1 \leqq r \leqq s \leqq n$, and

(4) $\sum_{J(I)} |G|<B$,

respectively.

If G is a function, then $G \in S_1$ on $[a, b]$ only if $\lim_{x,y \to p^+} G(x, y)$ and $\lim_{x,y \to p^-} G(x, y)$ exist and are zero for $p \in [a, b]$, and $G \in S_2$ on $[a, b]$ only if $\lim_{x \to p^+} G(p, x)$ and $\lim_{x \to p^-} G(x, p)$ exist for $p \in [a, b]$. Further, $G \in OL^{\circ}$ on $[a, b]$ only if $\lim_{x,y \to p^+} G(x, y)$, $\lim_{x,y \to p^-} G(x, y)$, $\lim_{x \to p^+} G(p, x)$ and $\lim_{x \to p^-} G(x, p)$ exist for $p \in [a, b]$. See B. W. Helton [**2**] and J. S. MacNerney [**7**] for additional details.

LEMMA 1.1. *If F and G are functions from $R \times R$ to R such that $F \in OP^{\circ}$ on $[a, b]$ and $G \in OB^{\circ}$ on $[a, b]$, then $F+G \in OP^{\circ}$ on $[a, b]$.*

Lemma 1.1 is part of a previous result by the author [**5**, Theorem 1].

LEMMA 1.2. *If G is a function from $R \times R$ to R such that $\int_a^b G$ exists, then $G \in OA^{\circ}$ on $[a, b]$.*

Lemma 1.2 is due to A. Kolmogoroff [**6**, p. 669]. The reader is also referred to results by W. D. L. Appling [**1**, Theorems 1, 2, p. 155] and B. W. Helton [**2**, Theorem 4.1, p. 304].

LEMMA 1.3. *If G is a function from $R \times R$ to R such that $G \in OB^{\circ}$ on $[a, b]$, then the following statements are equivalent:*

(1) *$G \in OM^{\circ}$ on $[a, b]$,*

(2) *$G \in OA^{\circ}$ on $[a, b]$, and*

(3) *$\int_a^b G$ exists.*

B. W. Helton [**2**, Theorem 3.4, p. 301] shows that (1) and (2) are equivalent. Further, by Lemma 1.2, (2) and (3) are equivalent.

LEMMA 1.4. *If F and G are functions from $R \times R$ to R such that $F \in OM^{\circ}$, OP° and $S_1 \cap S_2$ on $[a, b]$ and $G \in OM^{\circ}$ and OB° on $[a, b]$, then $F+G \in OM^{\circ}$ on $[a, b]$.*

Lemma 1.4 is proved in a previous paper by the author [**5**, Theorem 2]. In the original version [**5**, Theorem 2] the theorem is stated with the requirement that $\int_a^b G$ exist rather than $G \in OM^{\circ}$ on $[a, b]$. However, Lemma 1.3 establishes the equivalence of the two forms.

LEMMA 1.5. *If E is a finite set of points from $[a, b]$ and F, G and H are functions from $R\times R$ to R such that*

(1) *$G \in OB^\circ$ and S_2 on $[a, b]$,*
(2) *$H \in OP^\circ$ and $S_1 \cap S_2$ on $[a, b]$,*
(3) *$H+G \in OM^\circ$ on $[a, b]$, and*
(4) *$F \in S_2$ on $[a, b]$ and if $a \leqq x < y \leqq b$, then $F(x, y) = H(x, y)$ if $x \notin E$ and $y \notin E$,*

then $F+G \in OM^\circ$ on $[a, b]$.

PROOF. Lemma 1.1 establishes that $H+G \in OP^\circ$ on $[a, b]$. Further, $H+G \in S_1 \cap S_2$ on $[a, b]$. Let H' be the function defined on $[a, b]$ such that if $a \leqq x < y \leqq b$, then

(1) $H'(x, y) = 0$ if $x \notin E$ and $y \notin E$, and
(2) $H'(x, y) = F(x, y) - H(x, y)$ if $x \in E$ or $y \in E$.

Thus, $H' \in OM^\circ$ and OB° on $[a, b]$. Therefore, by Lemma 1.4, $H+G+H'$ is in OM° on $[a, b]$. Hence, since $H+G+H' \equiv F+G$ on $[a, b]$, $F+G \in OM^\circ$ on $[a, b]$.

LEMMA 1.6. *If G is a bounded function from $R\times R$ to R such that ${}_a\prod^b (1+G)$ exists and is not zero, then $G \in OP^\circ$ and OQ° on $[a, b]$.*

Lemma 1.6 is a special case of a previous result by the author [**4**, Theorem 2].

LEMMA 1.7. *If G is a bounded function from $R\times R$ to R such that $G \in OM^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, then $G \in OP^\circ$ and OQ° on $[a, b]$.*

PROOF. Since $G \in OM^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, ${}_a\prod^b (1+G)$ exists and is not zero. Therefore, it follows from Lemma 1.6 that $G \in OP^\circ$ and OQ° on $[a, b]$.

LEMMA 1.8. *If G is a function from $R\times R$ to R such that $G \in OB^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, then $G \in OQ^\circ$ on $[a, b]$.*

Lemma 1.8 is a special case of a previous result by the author [**5**, Theorem 3].

LEMMA 1.9. *If G is a bounded function from $R\times R$ to R such that ${}_a\prod^b (1+G)$ exists and is not zero, then $G \in OM^\circ$ on $[a, b]$.* .

Lemma 1.9 follows from Lemma 1.6 and a result of B. W. Helton [**2**, Theorem 4.2, p. 305].

LEMMA 1.10. *If G is a function from $R\times R$ to R such that $G \in OB^\circ$ and S_2 on $[a, b]$ and ${}_x\prod^y (1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^\circ$ on $[a, b]$.*

PROOF. Let $\varepsilon>0$. Since $G\in OB^{\circ}$ on $[a, b]$, there exist a subdivision D_0 of $[a, b]$ and a number $B>1$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_0 and $1\leqq r\leqq s\leqq n$, then

$$\left|\prod_{i=r}^{s}(1+G_i)\right|<B.$$

There exists a subdivision $E=\{w_q\}_{q=0}^t$ of $[a, b]$ such that if $1\leqq q\leqq t$ and $w_{q-1}<x<y<w_q$, then $|G(x,y)|<\frac{1}{2}$. Further, there exist sequences $\{u_q\}_{q=1}^t$ and $\{v_q\}_{q=1}^t$ such that

(1) $w_{q-1}<u_q<v_q<w_q$,

(2) if $w_{q-1}<x<y\leqq u_q$, then

$$|G(w_{q-1}, x)-G(w_{q-1}, y)|<\varepsilon(8t)^{-1},$$

(3) if $w_{q-1}<x<u_q$ and J is a subdivision of $[x, u_q]$, then

$$\sum_{J(I)}|G|<\varepsilon(8B^3t)^{-1},$$

(4) if $u_q\leqq x<y<w_q$, then

$$|G(x, w_q)-G(y, w_q)|<\varepsilon(8t)^{-1},$$

and

(5) if $v_q<x<w_q$ and J is a subdivision of $[v_q, x]$, then

$$\sum_{J(I)}|G|<\varepsilon(8B^3t)^{-1}.$$

We know from the hypothesis that ${}_{u_q}\prod^{v_q}(1+G)$ exists for $1\leqq q\leqq t$. Further, it follows from Lemma 1.8 that each of these integrals is nonzero. Thus, Lemma 1.9 implies that $G\in OM^{\circ}$ on $[u_q, v_q]$. Hence, for $1\leqq q\leqq t$, there exists a subdivision D_q of $[u_q, v_q]$ such that if $J=\{x_i\}_{i=0}^n$ is a refinement of D_q and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n}\left|1+G_i-\prod_{j=1}^{n(i)}(1+G_{ij})\right|<\varepsilon(8t)^{-1}.$$

Let D denote the subdivision $\bigcup_{q=0}^t D_q\cup E$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Let $\{x_{w(i)}\}_{i=0}^t$ be the subsequence of $\{x_i\}_{i=0}^n$ such that $x_{w(i)}=w_i$. Further, let $\{x_{u(i)}\}_{i=1}^t$ and $\{x_{v(i)}\}_{i=1}^t$ be the subsequences of $\{x_i\}_{i=0}^n$ such that $x_{u(i)}=u_i$ and $x_{v(i)}=v_i$. Let $T(q)$, $U(q)$ and $V(q)$ denote $\{i\}_{i=u(q)+1}^{v(q)}$, $\{i\}_{i=w(q-1)+1}^{u(q)}$ and $\{i\}_{i=v(q)+1}^{w(q)}$, respectively. Further, let U, V, $U'(q)$ and $V'(q)$ denote $\{w(i)+1\}_{i=0}^{t-1}$, $\{w(i)\}_{i=1}^{t}$, $\{i\}_{i=w(q-1)+2}^{u(q)}$ and $\{i\}_{i=v(q)+1}^{w(q)-1}$, respectively. Finally, let $S(q)$ and $S'(q)$ represent $U(q)\cup V(q)$ and $U'(q)\cup V'(q)$, respectively.

For $1 \leqq i \leqq n$, there exists a subdivision $\{x_{ij}\}_{j=0}^{n(i)}$ of $[x_{i-1}, x_i]$ such that

$$\left|\prod_{j=1}^{n(i)} (1+G_{ij}) - {}_{x_{i-1}}\prod^{x_i} (1+G)\right| < \varepsilon(8n)^{-1}.$$

Let H_i represent $1+G_i-\prod_{j=1}^{n(i)} (1+G_{ij})$. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} &\left|1+G_i - {}_{x_{i-1}}\prod^{x_i}(1+G)\right| < \sum_{i=1}^{n} |H_i| + [\varepsilon(8n)^{-1}]n \\
&= \sum_{q=1}^{t}\sum_{i\in S(q)} |H_i| + \sum_{q=1}^{t}\sum_{i\in T(q)} |H_i| + \frac{\varepsilon}{8} \\
&< \sum_{q=1}^{t}\sum_{i\in S(q)} |H_i| + [\varepsilon(8t)^{-1}]t + \frac{\varepsilon}{8} \\
&= \sum_{i\in U\cup V} |H_i| + \sum_{q=1}^{t}\sum_{i\in S'(q)} |H_i| + \frac{\varepsilon}{4} \\
&= \sum_{i\in U\cup V} |H_i| \\
&\quad + \sum_{q=1}^{t}\sum_{i\in S'(q)} \left|\{1+G_i\} - \left\{1 + \sum_{j=1}^{n(i)} \left[\prod_{r=1}^{j-1}(1+G_{ir})\right]\right.\right. \\
&\qquad\qquad \left.\left. \cdot [G_{ij}]\left[\prod_{s=j+1}^{n(i)}(1+G_{is})\right]\right\}\right| + \frac{\varepsilon}{4} \\
&\leqq \sum_{i\in U\cup V} |H_i| + \sum_{q=1}^{t}\sum_{i\in S'(q)} \left[|G_i| + B^2\sum_{j=1}^{n(i)} |G_{ij}|\right] + \frac{\varepsilon}{4} \\
&< \sum_{i\in U\cup V} |H_i| + [\varepsilon(8B^3t)^{-1}]t + B^2[\varepsilon(8B^3t)^{-1}]t + \frac{\varepsilon}{4} \\
&< \sum_{i\in U\cup V} |H_i| + \frac{\varepsilon}{2} \\
&\leqq \sum_{i\in U} |(1+G_i) - (1+G_{i1})| + \sum_{i\in U} |1+G_{i1}| \left|-1 + \prod_{j=2}^{n(i)}(1+G_{ij})\right| \\
&\quad + \sum_{i\in V} |(1+G_i) - (1+G_{i,n(i)})| \\
&\quad + \sum_{i\in V} |1+G_{i,n(i)}| \left|-1 + \prod_{j=1}^{n(i)-1}(1+G_{ij})\right| + \frac{\varepsilon}{2} \\
&\leqq \sum_{i\in U} |G_i - G_{i1}| + B\sum_{i\in U}\left|-1 + \left\{1 + \sum_{j=2}^{n(i)}\left[\prod_{r=2}^{j-1}(1+G_{ir})\right][G_{ij}]\right.\right. \\
&\qquad\qquad \left.\left. \cdot \left[\prod_{s=j+1}^{n(i)}(1+G_{is})\right]\right\}\right|
\end{aligned}$$

$$+\sum_{i\in V}|G_i - G_{i,n(i)}| + B\sum_{i\in V}\left|-1 + \left\{1 + \sum_{j=1}^{n(i)-1}\left[\prod_{r=1}^{j-1}(1+G_{ir})\right][G_{ij}] \cdot \left[\prod_{s=j+1}^{n(i)-1}(1+G_{is})\right]\right\}\right| + \frac{\varepsilon}{2}$$

$$< [\varepsilon(8t)^{-1}]t + B^3\sum_{i\in U}\sum_{j=2}^{n(i)}|G_{ij}| + [\varepsilon(8t)^{-1}]t + B^3\sum_{i\in V}\sum_{j=1}^{n(i)-1}|G_{ij}| + \frac{\varepsilon}{2}$$

$$< B^3[\varepsilon(8B^3t)^{-1}]t + B^3[\varepsilon(8B^3t)^{-1}]t + \frac{3\varepsilon}{4}$$

$$= \varepsilon.$$

Therefore, $G \in OM^\circ$ on $[a, b]$.

Lemma 1.10 is not true if only ${}_a\prod^b (1+G)$ is required to exist rather than ${}_x\prod^y (1+G)$ for $a \leqq x < y \leqq b$. For example, consider the function G defined on $[0, 1]$ such that, for $0 \leqq x < y \leqq 1$,

(1) $G(0, x) = -1$,

(2) $G(x, y) = y - x$ if $x \neq 0$ and y is irrational, and

(3) $G(x, y) = x - y$ if $x \neq 0$ and y is rational.

Thus, $G \in OB^\circ$ and S_2 on $[a, b]$ and ${}_a\prod^b (1+G)$ exists and is zero. However, ${}_x\prod^y (1+G)$ does not exist for $a \leqq x < y \leqq b$, and thus, $G \notin OM^\circ$ on $[a, b]$.

LEMMA 1.11. *If H and G are functions from $R\times R$ to R such that $H \in OL^\circ$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and either $G \in OM^\circ$ on $[a, b]$ or $G \in OA^\circ$ on $[a, b]$, then $HG \in OM^\circ$ and OA° on $[a, b]$.*

Lemma 1.11 is a modification of a result of B. W. Helton [3, Theorem 2, p. 494] obtained by using Lemma 1.3.

THEOREM 1. *If F and G are functions from $R\times R$ to R such that $F \in OP^\circ$ and $S_1 \cap S_2$ on $[a, b]$ and $G \in OB^\circ$ and S_2 on $[a, b]$, then any two of the following statements imply the other:*

(1) *$F+G \in OM^\circ$ on $[a, b]$,*

(2) *$F \in OM^\circ$ on $[a, b]$, and*

(3) *$G \in OM^\circ$ on $[a, b]$.*

PROOF (1, 2→3). There exists a subdivision $E=\{w_i\}_{i=0}^t$ of $[a, b]$ such that if $1 \leqq i \leqq t$ and $w_{i-1} < x < y < w_i$, then $|F(x, y)| < \frac{1}{2}$. Let $F'(x, y) = F(x, y)$ if $x \notin E$ and $y \notin E$, and let $F'(x, y) = 0$ if $x \in E$ or $y \in E$. Thus, $(1+F')^{-1}$ is in OL° on $[a, b]$. Further, it follows from Lemma 1.5 that $F'+G \in OM^\circ$ on $[a, b]$ and $F' \in OM^\circ$ on $[a, b]$. Hence, since $F' \in OM^\circ$ on

$[a, b]$, Lemma 1.7 implies that $F' \in OQ^\circ$ on $[a, b]$. Also, note that $G(1+F')^{-1}$ is in OB° and OP° on $[a, b]$.

We now establish that ${}_x\prod^y [1+G(1+F')^{-1}]$ exists by using the Cauchy criterion for product integrals, where $a \leqq x < y \leqq b$. Let $\varepsilon > 0$. There exist a subdivision D of $[x, y]$ and positive numbers B and β such that if J and K are refinements of D, then

(1) $|\prod_{J(I)} (1+F')| > \beta$,

(2) $|\prod_{J(I)} [1+G(1+F')^{-1}]| < B$,

(3) $|\prod_{J(I)} (1+F') - \prod_{K(I)} (1+F')| < \beta\varepsilon(2B)^{-1}$, and

(4) $|\prod_{J(I)} (1+F'+G) - \prod_{K(I)} (1+F'+G)| < \beta\varepsilon/2$.

Suppose J and K are refinements of D. Thus,

$$\begin{aligned}
\frac{\beta\varepsilon}{2} &> \left|\prod_{J(I)} (1+F'+G) - \prod_{K(I)} (1+F'+G)\right| \\
&= \left|\left\{\prod_{J(I)} (1+F')\right\}\left\{\prod_{J(I)} [1+G(1+F')^{-1}]\right\}\right. \\
&\qquad \left. - \left\{\prod_{K(I)} (1+F')\right\}\left\{\prod_{K(I)} [1+G(1+F')^{-1}]\right\}\right| \\
&\geqq \left|\prod_{J(I)} (1+F')\right| \left|\prod_{J(I)} [1+G(1+F')^{-1}] - \prod_{K(I)} [1+G(1+F')^{-1}]\right| \\
&\qquad - \left|\prod_{J(I)} (1+F') - \prod_{K(I)} (1+F')\right| \left|\prod_{K(I)} [1+G(1+F')^{-1}]\right| \\
&> \beta\left|\prod_{J(I)} [1+G(1+F')^{-1}] - \prod_{K(I)} [1+G(1+F')^{-1}]\right| - [\beta\varepsilon(2B)^{-1}]B,
\end{aligned}$$

and hence,

$$\varepsilon > \left|\prod_{J(I)} [1+G(1+F')^{-1}] - \prod_{K(I)} [1+G(1+F')^{-1}]\right|.$$

Therefore, the desired product integral exists.

Now, since ${}_x\prod^y [1+G(1+F')^{-1}]$ exists for $a \leqq x < y \leqq b$ and $G(1+F')^{-1}$ is in OB° on $[a, b]$, it follows from Lemma 1.10 that $G(1+F')^{-1}$ is in OM° on $[a, b]$. Hence, since $1+F'$ is in OL°, Lemma 1.11 implies that $G \in OM^\circ$ on $[a, b]$.

PROOF $(2, 3 \to 1)$. This result is stated as Lemma 1.4 and is proved in a previous paper by the author [5, Theorem 2].

PROOF $(1, 3 \to 2)$. It follows from Lemma 1.1 that $F+G \in OP^\circ$ on $[a, b]$. Further, $F+G \in S_1 \cap S_2$ on $[a, b]$, and $-G \in OB^\circ$ and OM° on $[a, b]$. Therefore, Lemma 1.4 implies that $F \equiv F+G-G$ is in OM° on $[a, b]$.

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