CONTINUITY OF MAXIMAL MONOTONE SETS IN BANACH SPACE[1]

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ABSTRACT. A monotone set-valued map of a Banach space to its dual is shown to map line segments into bounded sets. It follows that convergent sequences are mapped into bounded sets and, when the space is separable or reflexive, this imposes continuity requirements on maximal monotone maps.

Let X be a (real) Banach space with dual X^* and denote the value of $w \in X^*$ on $x \in X$ by (w, x). We consider a subset T of $X \times X^*$ whose image $T(x) = \{w \in X^* : [x, w] \in T\}$ at each $x \in X$ is nonempty. The set T is called *monotone* if

$$(w_1 - w_2, x_1 - x_2) \geqq 0, \qquad [x_1, w_1], [x_2, w_2] \in T,$$

and *maximal monotone* if it is not properly included in a monotone set. We give an elementary proof that monotone sets map line segments into bounded sets and then deduce some continuity properties of maximal monotone sets.

THEOREM 1. *Let T be a monotone subset of $X \times X^*$ with nonempty image $T(x)$ for each $x \in X$. Let x_0 and x_1 belong to X, and set $x_t = x_0 + t(x_1 - x_0)$ for $t \in [0, 1)$. Then $\bigcup \{T(x_t) : t \in [0, 1)\}$ is bounded in X^*.*

PROOF. Let $v \in X$ and choose $w_1 \in T(x_1)$, $w_2 \in T(x_0 + v)$. For each $u \in T(x_t)$ we have by monotonicity of T

$$(w_1 - u, x_1 - x_t) = (1 - t)(w_1 - u, x_1 - x_0) \geqq 0,$$

so we obtain $(u, x_1 - x_0) \leqq (w_1, x_1 - x_0)$. The monotonicity also gives us

$$(u, v) \leqq t(u, x_1 - x_0) + (w_2, v - t(x_1 - x_0))$$
$$\leqq |(w_1, x_1 - x_0)| + (w_2, v) + |(w_2, x_1 - x_0)|.$$

Repeating the argument with v replaced by $-v$ shows that (u, v) is bounded independently of $t \in [0, 1)$ and $u \in T(x_t)$, so the desired result follows by uniform boundedness.

THEOREM 2. *Let T be a monotone set with $T(x)$ nonempty for each $x \in X$. If $\{x_n\}$ is a convergent sequence in X, then $\bigcup \{T(x_n): n \geq 1\}$ is bounded in X^*.*

PROOF. Since $T(x_n)$ is bounded for each $n \geq 1$ by Theorem 1, it suffices to show that for any choice of $v_n \in T(x_n)$ for each n, the sequence $\{v_n\}$ is bounded. But this will follow from the proof in [3].

REMARK. The writer was informed by the referee and by L. Nirenberg of the references [4], [5] which contain Theorem 2. The simple proof of Theorem 1 seems to be new, and it extends easily to the case where the domain $D(T) = \{x \in X: T(x)$ is nonempty$\}$ is not all of X. One needs then to require that x_0 belong to the interior of $D(T)$ and $x_1 \in D(T)$. Theorem 2 holds if $\{x_n\}$ converges to a point in the interior of $D(T)$.

A subset of $X \times X^*$ is called *vaguely continuous* [1] if for each pair $x_0, x_1 \in X$ there exists a null sequence of positive real numbers $\{t_n: n \geq 1\}$, a sequence $v_n \in T(x_n)$, where $x_n = x_0 + t_n(x_1 - x_0)$, and a $v_0 \in T(x_0)$ such that $\{v_n\}$ is weak* convergent to v_0. We define T to be *hemicontinuous* if, for each pair $x_0, x_1 \in X$, for every null sequence of positive real numbers $\{t_n\}$, and for every choice of $v_n \in T(x_n)$, x_n given as above, there exists a subsequence $\{v_n'\}$ which is weak* convergent to some $v_0 \in T(x_0)$. Finally, we define T to be *demicontinuous* if for every sequence $\{x_n\}$ which converges to $x_0 \in X$, and for every choice of $v_n \in T(x_n)$, there exists a subsequence $\{v_n'\}$ which is weak* convergent to some $v_0 \in T(x_0)$.

It is clear that each of these conditions implies the preceding one. Also, if $T(x_0) = \{v_0\}$ is a singleton in the definition of either hemicontinuity or demicontinuity, then the original sequence $\{v_n\}$ converges to v_0. Hence, if T is a function, these definitions agree with the usual notions of hemicontinuity and demicontinuity [3].

It is known [1, Theorem 1.2] that if T is monotone, vaguely continuous, and if $T(x)$ is closed and convex for each $x \in X$, then T is maximal monotone. Conversely [1, Lemma 1.1], if T is maximal monotone it follows that $T(x)$ is closed and convex for each $x \in X$. The converse is complete when the closed unit ball of X^* is weak* sequentially compact.

THEOREM 3. *Let X be separable or reflexive. A maximal monotone set T with $T(x)$ nonempty for each $x \in X$ is demicontinuous.*

PROOF. Let $\{x_n\}$ converge to x_0 in X and $v_n \in T(x_n)$. Theorem 2 asserts that $\{v_n\}$ is bounded, so by weak* compactness of the unit ball of X^*, it suffices to show that if $\{v_n\}$ is weak* convergent to v_0, then $v_0 \in T(x_0)$. But for each $x \in X$ and $w \in T(x)$ we have by monotonicity

$$(v_n - w, x_n - x) \geq 0, \qquad n \geq 1,$$

so taking the limit gives

$$(v_0 - w, x_0 - x) \geqq 0, \qquad [w, x] \in T.$$

Since T is maximal monotone, $v_0 \in T(x_0)$.

COROLLARY 1. *Let X be separable or reflexive. The set T is maximal monotone if and only if it is monotone, vaguely continuous, and $T(x)$ is closed and convex for each $x \in X$.*

Clearly, Corollary 1 is true with vague continuity replaced by hemi-continuity or demicontinuity (or upper semicontinuity [2]) so these notions agree for monotone sets whose images are closed and convex. Furthermore, since the sum

$$T_1 + T_2 = \{[x, w_1 + w_2]: [x, w_j] \in T_j, j=1, 2\}$$

of a vaguely continuous set T_1 and a hemicontinuous set T_2 is vaguely continuous, we have a perturbation result.

COROLLARY 2. *Let T_1 be vaguely continuous and X be separable or reflexive. If T_2 is maximal monotone and the sum $T_3 = T_1 + T_2$ is monotone with $T_3(x)$ closed, convex and nonempty for each $x \in X$, then T_3 is maximal monotone. In particular, if T_2 and T_3 are monotone with $T_2(x)$ and $T_3(x)$ closed, convex and nonempty for each $x \in X$, and if T_1 is vaguely continuous and single valued, then T_2 is maximal monotone if and only if T_3 is maximal monotone.*

Finally, if the range of T, $\bigcup \{T(x): x \in X\}$ is all of X^* and X is reflexive, then the inverse set $T^{-1} = \{[x, w]: [w, x] \in T\}$ is monotone (maximal monotone) if and only if T is monotone (respectively, maximal monotone). This gives

COROLLARY 3. *Let T be a monotone set in $X \times X^*$ with X reflexive. Suppose that $T(x)$ and $T^{-1}(w)$ are closed, convex and nonempty for each $x \in X$ and $w \in X^*$. Then T is vaguely continuous (or hemicontinuous or demicontinuous) if and only if T^{-1} is vaguely continuous (or hemicontinuous or demicontinuous).*

The preceding results give improvements in many of the results of [1], [2]: we obtain converses to certain of these, and we may drop bounded-ness assumptions in others. Theorem 2 shows that monotone sets map compact sets into bounded sets, so if X has finite dimension, a monotone T maps bounded sets to bounded sets and is maximal if and only if it is continuous.

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