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STEINITZ CLASSES IN QUARTIC FIELDS

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ABSTRACT. Let K be normal quartic over the rationals. Let $l \equiv 3$ (4) be an odd prime. If the class number of K is even, there is a normal extension L of degree l over K such that the relative discriminant is principal, but L has no relative integral base over K.

I. **Introduction and results.** Let K be an algebraic number field, and L a finite extension. The relative discriminant $D_{L/K}$ is an ideal of K. Let d be the discriminant of a K-base of L and (d) the principal ideal generated by d. Then $D_{L/K}=B^2(d)$ for some fractional ideal B of K. The ideal class to which B belongs is written $C(L/K)$ and is called the Steinitz class of L with respect to K.

Artin [1] showed that L has a relative integral base over K if and only if $C(L/K)$ is principal. Thus if the class number h_K is odd, L has a relative integral base if and only if $D_{L/K}$ is principal.

The story is different if h_K is even; $C(L/K)$ may be in a class of order 2, i.e., $D_{L/K}$ can be principal without L having an integral K-base.

Fröhlich [2] showed that every ideal class of K is a Steinitz class for some quadratic extension. For a fixed odd prime l, Long [5] found which classes of K can be Steinitz classes for some normal extension of degree l. We repeat his result. The classes are those of the form $C^{l-1/2}$, where C is a class containing a prime divisor of l or C contains a prime which splits fully upon adjunction of the lth roots of unity.

Let K be an algebraic number field, and let l be an odd prime. We say K has property $(*)$ with respect to l if there is a normal extension L of degree l which has no relative integral base, but $D_{L/K}$ is principal.

No field K with odd class number can have $(*)$ with respect to any prime; $D_{L/K}$ is principal if and only if L has a relative integral base. Thus, for the rest of the paper, we only deal with fields K for which h_K is even.

If $l \equiv 1$ (mod 4) and $h_K=2$, it is clear that K does not have $(*)$ with respect to l. For the case $l \equiv 3$ (4) and h_K even, the problem seems harder. We do not know of any such fields which do not have property $(*)$ with respect to l.

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THEOREM 1. *Let K be quadratic over the rationals Q. Suppose h_K i even and $l \equiv 3$ (4) is prime. Then K has $(*)$ with respect to l.*

THEOREM 2. *Let K be normal quartic over Q. Suppose h_K is even and $l \equiv 3$ (4) is prime. Then K has $(*)$ with respect to l.*

II **Proofs.** First, some preliminary remarks. The ideal classes of K which are Steinitz classes for some normal extension of K of degree l form a group [5]. If K does not have $(*)$ then all primes which split fully upon adjunction of an lth root of unity ζ are in classes of odd order. Thus the 2-part of the Hilbert class field of K lies in $K(\zeta)$ and hence $h_K \equiv 2 \pmod 4$. Also $K((-l)^{1/2})$ is quadratic unramified over K and K is totally imaginary.

Theorem 1 is easy to complete. We have K imaginary and $l \mid D_{K/Q}$. Now $K \neq Q((-l)^{1/2})$, since $-l$ is not a square in K; thus l is the square of a prime ideal in a class of order 2.

We divide Theorem 2 into two cases. First assume K is cyclic over Q. Let k be the unique subfield; k is real. A prime fully ramified from Q to K is $\equiv 1$ (4) or is 2. Any prime ramified in k is fully ramified in K. Thus l is ramified from k to K.

Let h_0 be the narrow class number of k. Let t be the number of primes (including infinite primes) ramifying from k to K. By a formula of Hasse [3, p. 99], the number h of ambiguous classes of K over k is

$$h = h_0 2^{t+q^*-3} \tag{1}$$

and the number h' of ambiguous classes of K containing ambiguous ideals is

$$h' = h_0 2^{t+q-3} \tag{2}$$

where q^*, q are given by

$$2^{q^*} = (E_k \cap N_{K/k}K^* : E_k^2), \tag{3}$$

$$2^q = (E_k \cap N_{K/k}E_K : E_k^2). \tag{4}$$

In (3), (4), E_K, E_k are the unit groups.

The ambiguous classes of K are a group, and since $h_K \equiv 2$ (4), we also have $h \equiv 2$ (4). In the case of K cyclic over Q, we have $t \geqq 4$ and hence h_0 is odd and $q^* = 0$. Thus $k = Q(p^{1/2})$ or $Q(2^{1/2})$ and $p \equiv 1$ (4). Now l is inert in k; otherwise $t \geqq 5$. Thus $D_{K/k} = (lp^{1/2})$. It follows that l is the square of a prime in the class of order 2 in K.

Next, let K have Galois group $C_2 \times C_2$. Let k be the real subfield. Suppose l ramifies from Q to k. Then $2 \mid h_0$ and the fundamental unit ε of k has norm 1. Hasse's formula yields $t=3$, $q^*=0$; $t=2$, $q^*=1$; or $t=2$,

$q^*=0$. Since ε is totally positive, ε is a norm at all primes except possibly one; hence ε is a global norm and $q^*\geqq 1$. Our only alternative is $t=2$, $q^*=1$, $h_0\equiv 2$ (4). Then K must be $k((-l)^{1/2})$ which contradicts the fact that $-l$ is not a square in K.

Finally, suppose l does not ramify in k. In (1), $t\geqq 3$ and our alternatives are $t=3$, $q^*=0$; $t=3$, $q^*=1$; $t=4$, $q^*=0$. If $q^*=1$, $t=3$, then h_0 is odd and $k=Q(p^{1/2})$, $p\equiv 1$ (4) a prime or $p=2$. In either case, ε is not totally positive, so $q^*\neq 1$.

In the other cases, $q^*=q=0$. If $t=3$, l is the only finite prime ramifying, so l ramifies in the class of order 2. If $t=4$, h_0 is odd and $k=Q(p^{1/2})$ or $Q(2^{1/2})$ as before. Then $k'=Q((-lr)^{1/2})$ is a subfield of K, where r is a prime different from p. Thus l, r are inert in k and ramify from k to K. Hence the prime in K dividing l must lie in the class of order 2.

III. **Additional remarks.** It is clear that any normal extension K of Q with even class number must have property $(*)$ with respect to any odd prime l, simply because l cannot have even ramification index in K.

If $l\equiv 3$ (4) and K is an abelian field with even class number and not having property $(*)$, then the largest subfield of K which has degree a power of 2 also does not have $(*)$.

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